2. Fundamentals of Logic

Primitive statement:

a statement that can not be broken down into simpler form and it is either *true* or *false* but can not be both.

Example:

p: John is a student

q: UK is a university

Compound statement:

a statement that is formed of primitive statements with *logical connectives* such as

1. Negation: $\bar{p}$ (or, $\neg p$)
2. Conjunction: $p \wedge q$ (p and q)
3. Disjunction: $p \vee q$ (p or q)
4. Implication: $p \rightarrow q$ (p implies q)
5. Equivalence: $p \longleftrightarrow q$ (p if and only if q)

"If $x = 3$ then $x^2 = 9$" is a *true* statement

"If $x = 5$ then $x + 2 = 11$" is a *false* statement

Truth Table: 1: *true* 0: *false*

p	$\bar{p}$
0	1
1	0

p	q	$p \land q$	$p \lor q$	$p \to q$	$p \leftrightarrow q$
0	0	0	0	1	1
0	1	0	1	1	0
1	0	0	1	0	0
1	1	1	1	1	1

Note: Two statements are equivalent if their truth tables are the same.

Note: $(p \rightarrow q) \longleftrightarrow (\bar{p} \vee q)$

$p \rightarrow q$ (not ture only if the implication is violated)

If the hypothesis is not satisfied, then no matter what truth value q has, the implication is not violated

Tautology: a statement that is always true

$$p \vee \bar{p}$$

Contradiction: a statement that is always false

$$p \wedge \bar{p}$$

Notations:

T_o: tautology

F_o: contradiction

Theorem: a (mathematical) statement that can be shown to be true.

Example: $p \vee \bar{p}$ is a tautology

p	$\bar{p}$	$p \vee \bar{p}$
0	1	1
1	0	1

$p \rightarrow (p \vee q)$ is a tautology

p	q	$p \vee q$	$p \rightarrow (p \vee q)$
0	0	0	1
0	1	1	1
1	0	1	1
1	1	1	1

<u>Logical Equivalence: The Laws of Logic</u>:

- use truth tables and propositions to determine when two statements are functionally equivalent

<u>Def:</u> Two statements s_1 and s_2 are said to be logically equivalent, denoted $s_1 \Longleftrightarrow s_2$, when the truth tables for s_1 and s_2 are the same.

<u>Examples:</u>

$$(p \rightarrow q) \Longleftrightarrow (\bar{p} \vee q)$$

$$(p \leftrightarrow q) \Longleftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$$

DeMorgan's Law

$$\overline{p \wedge q} \Longleftrightarrow \bar{p} \vee \bar{q}$$

$$\overline{p \vee q} \Longleftrightarrow \bar{p} \wedge \bar{q}$$

Distributive Law

$$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$$

$$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$$

Commutative Law

$$p \vee q \Leftrightarrow q \vee p$$

$$p \wedge q \Leftrightarrow q \wedge p$$

Associative Law

$$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$$

$$p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$$

Idempotent Law

$$p \vee p \Leftrightarrow p \;; \qquad p \wedge p \Leftrightarrow p$$

Idnetity Law

$$p \vee F_o \Leftrightarrow p \;; \qquad p \wedge T_o \Leftrightarrow p$$

Inverse Law

$$p \mathbf{V} \bar{p} \Leftrightarrow T_o \; ; \qquad p \wedge \bar{p} \Leftrightarrow F_o$$

Domination Law

$$p \mathbf{V} T_o \Leftrightarrow T_o \; ; \qquad p \wedge F_o \Leftrightarrow F_o$$

Absorpion Law

$$p \mathbf{V} (p \wedge q) \Leftrightarrow p \; ; \qquad p \wedge (p \mathbf{V} q) \Leftrightarrow p$$

Law of Double Negation

$$\bar{\bar{p}} \Leftrightarrow p$$

Example: Prove the following is true $(\bar{p} \mathbf{V} q) \wedge (p \wedge (p \wedge q)) \Leftrightarrow (p \wedge q)$

$$LHS \Leftrightarrow (\bar{p} \mathbf{V} q) \wedge ((p \wedge p) \wedge q)$$

$$\Leftrightarrow (\bar{p} \mathbf{V} q) \wedge (p \wedge q)$$

$$\Leftrightarrow (\bar{p} \wedge (p \wedge q) \mathbf{V} (q \wedge (p \wedge q))$$

$$\Leftrightarrow ((\bar{p} \wedge p) \wedge q) \mathbf{V} (p \wedge q)$$

$$\Leftrightarrow (F_o \wedge q) \vee (p \wedge q)$$

$$\Leftrightarrow F_o \vee (p \wedge q)$$

$$\Leftrightarrow p \wedge q$$

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s: statement involving only $\wedge$, $\vee$

s^d: dual of s, by replacing $\wedge$, $\vee$, T_o , F_o with $\vee$, $\wedge$, F_o T_o

Example:

s: $p \vee \bar{p}$

s^d: $p \wedge \bar{p}$

Theorem 2.1: (The Principle of Duality)

s, t: statements

If $s \Leftrightarrow t$ then $s^d \Leftrightarrow t^d$

Proof can be proved using Boolean algebra. However, we will not prove it until Chapter 15.

Subsitution Rule:

(1) P: compound statement and a tautology

If p is a statement that appears in P and we replace each occurence of p with the same statement q then the resulting compound statement Q is also a tautology

Proof. If p is replaced with q, then the same truth values can be applied to q as well. Since with these truth values, P is a tautology, Q must be a tautology too.

(2) P: compound statement

p: a statement appears in P

If q is a statement such that $p \Leftrightarrow q$ then if we replace each occurence of p in P with q then the resulting compound statement $Q \Leftrightarrow P$

Proof. Similar to the above case.

Example:

$$\text{Simplify} \quad \overline{(P \wedge q) \to r}$$

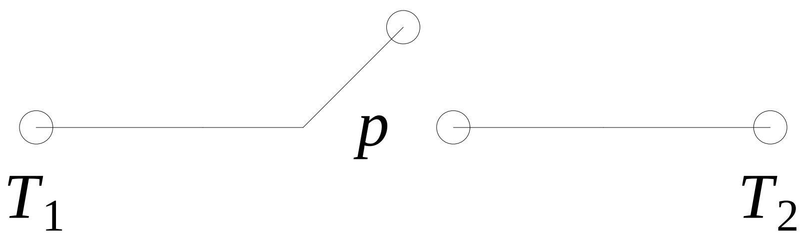
$$\Leftrightarrow \overline{\overline{(p \wedge q)} \vee r}$$

$$\Leftrightarrow \overline{\overline{(p \wedge q)}} \wedge \bar{r}$$

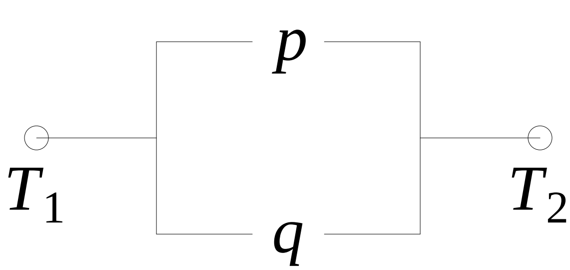
$$\Leftrightarrow (p \wedge q) \wedge \bar{r}$$

How to use logic laws to simplify logic circuits:

Example: Network with one switch:



Current flows from T_1 to T_2 if the switch p is closed.

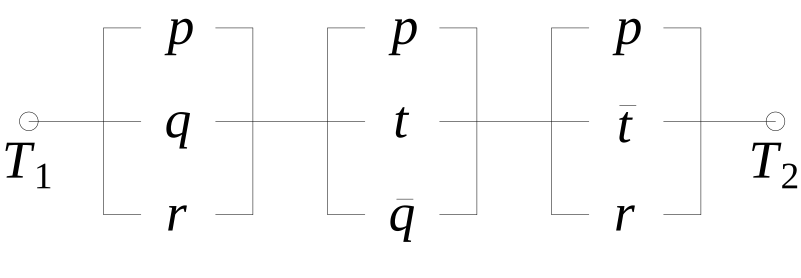


$p \vee q$

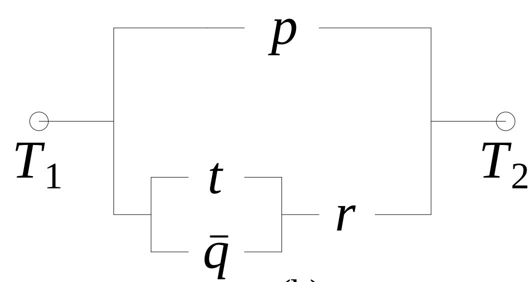
$p \wedge$

In the first case, current flows from T_1 to T_2 if either of the switches p, q is closed. In the second case, current flows from T_1 to T_2 if both switches are closed.

In the following, prove that (a) and (b) are equivalent.



(a)



(b)

$$(p \vee q \vee r) \wedge (p \vee t \vee \bar{q}) \wedge (p \vee \bar{t} \vee r)$$

$$\Leftrightarrow\ p \vee ((q \vee r) \wedge (t \vee \bar{q}) \wedge (\bar{t} \vee r))$$

$$\Leftrightarrow\ p \vee ((t \vee \bar{q}) \wedge ((q \wedge \bar{t}) \vee r))$$

$$\Leftrightarrow\ p \vee ((t \vee \bar{q}) \wedge (\overline{(\bar{q} \wedge t)} \vee r))$$

$$\Leftrightarrow\ p \vee (((t \vee \bar{q}) \wedge \overline{(\bar{q} \wedge t)}) \vee ((t \vee \bar{q}) \wedge r))$$

$$\Leftrightarrow\ p \vee (F_o \vee ((t \vee \bar{q}) \wedge r))$$

$$\Leftrightarrow\ p \vee ((t \vee \bar{q}) \wedge r)$$