11. Graph Theory

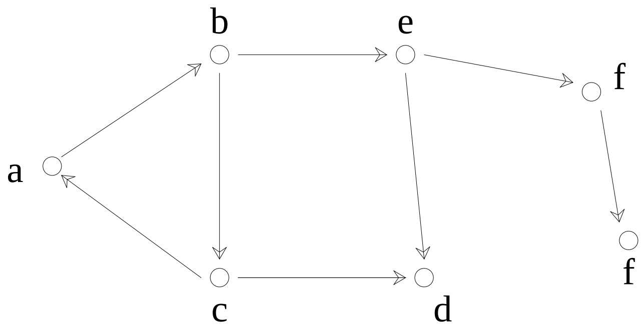
Def: V: non-empty vertex or node set

$E \subseteq V \otimes V$: edge set

$G \equiv (V, E)$: *directed graph* on V,

or *digraph* on V

Example:



V={a, b, c, d, e, f, g}

E={(a,b), (b,c), (c,a), ... }

Note: $(a, b) \neq (b, a)$

Def: When there is no concern about the direction of an edge, the graph is called *undirected*. In this case, an edge between two vertices a and b is represented by $\{a, b\}$.

Note: $\{a, b\} = \{b, a\}$

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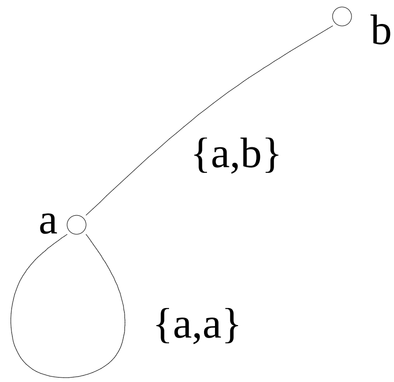


$\{a, a\}$ is a loop from a to a.

If a graph is not specified as directed or undirected, it is assumed to be undirected.

Def:

$G = (V, E)$: undirected graph

An $x - y$ *walk* is a finite alternating sequence

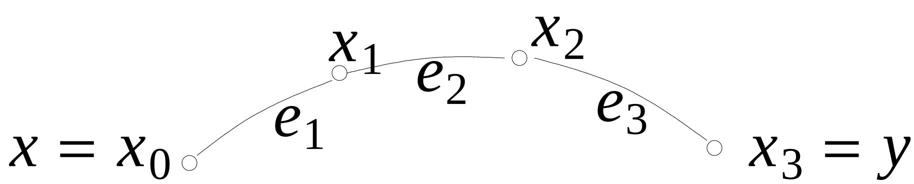
$$x = x_0,\ e_1,\ x_1,\ e_2,\ \ldots,\ e_{n-1},\ x_{n-1},\ e_n,\ x_n = y$$

of vertices and edges from G, starting at vertex x and ending at vertex y, with the n edges $e_i = \{x_{i-1}, x_i\}$, $1 \le i \le n$

length $= n =$ no of edges

If $n = 0$, a trivial walk

When $x = y$ and $n > 0$, called a *closed* walk, otherwise, *open* walk.

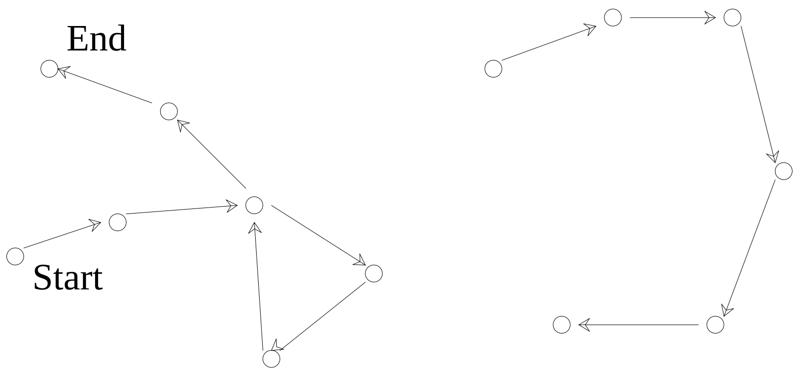


Def:

If no edge in an walk is repeated, then the walk is called a *trail*. A closed trail is called a *circuit*.

When no vertex of the $x - y$ walk occurs more than once, then the walk is called a *path*. A closed path is called a *cycle*.

(The term "cycle" will always imply the presence of at least 3 distinct edges)



Trail Path

For directed graphs, we put "directed" in front of all the terms defined above.

Theorem:

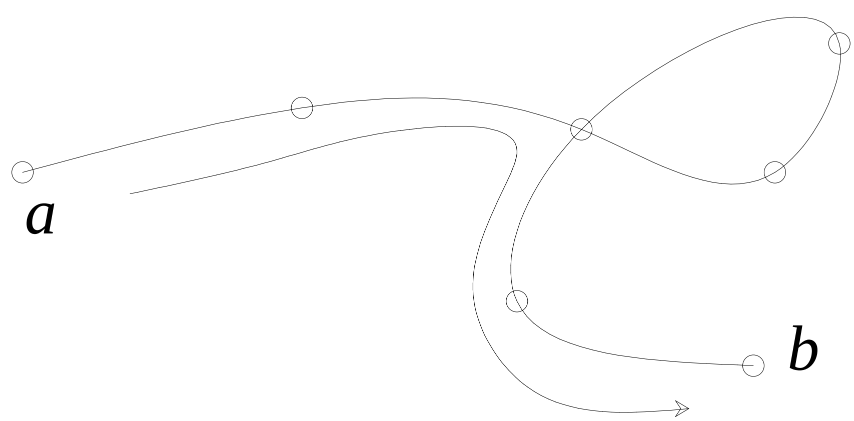
$G = (V, E)$: undirected graph

$a, b \in V, \ a \neq b$

If there exists a *trail* from a to b then there is a *path* from a to b.

Proof.

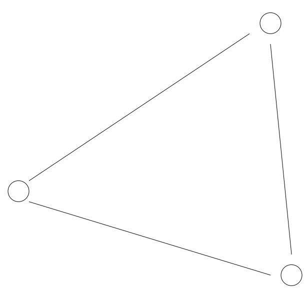
Let T be the set of all trails from a to b. T has an element with the smallest length. Let P be such a trail. Then P must be a path. Why?



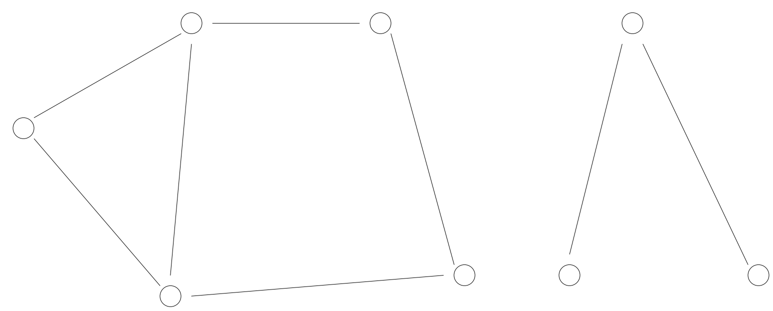
<u>Def:</u>

$G = (V, E)$: undirected graph

G is called *connected* if there is a path between any two distinct vertices of G.



Connected



Not connected

A digraph is said to be connected if the associated undirected graph is connected.

<u>Def:</u>

$G = (V, E)$ (directed or undirected)

A graph $G_1 = (V_1, E_1)$ is called a *subgraph* of G if $V_1 \subseteq V$ and $E_1 \subseteq E$

(edges in E_1 must be incident with vertices in V_1)

<u>Def:</u>

$G = (V, E)$ (directed or undirected)

A connected subgraph of G is said to be a *component* of G if it is not properly contained in any connected subgraph of G.

<u>Notation:</u>

The number of components of G is denoted by $K(G)$.

<u>Note:</u>

A graph is connected iff $K(G) = 1$

<u>Def:</u>

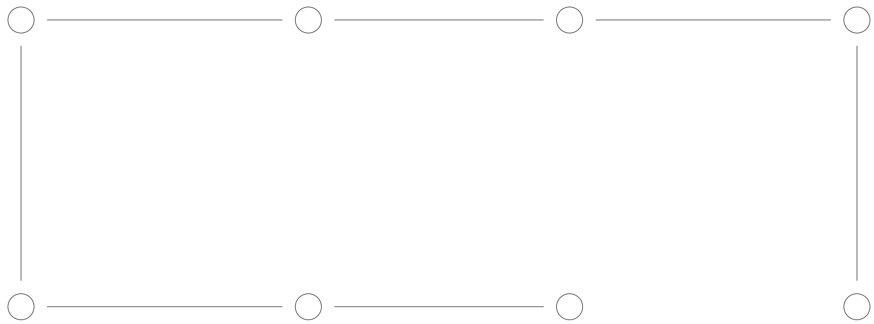
A graph $G = (V, E)$ is called a *multigraph* if $\exists$ $a, b \in V, a \neq b$, such that there are two or more edges between a and b (undirected) (from a to b (directed)).

<u>Def:</u>

multiplicity $= 3$

n –graph: no vertex has multiplicity $> n$

Example: (No. 8, page 412)



Removing any edge would result in a disconnected graph

Can such a graph be characterized using the concept of graph path? (the entire graph is a path)

Example: (No. 9, page 412)

If a graph satisfies the above condition then

- it must be loop-free
- G can not be a multigraph
- If G has n vertices then it must have $n - 1$ edges

Example: (No. 10, page 412)

a)

If $G = (V, E)$: undirected, $|V| = m$, $|E| = n$, and no loop, then $2n \le m^2 - m$

b)

Since $(v_1, v_2) \ne (v_2, v_1)$ for digraph without loops, we have $n \le m^2 - m$

11.2 Complements and Graph Isomorphism

- study the structure of graphs

Def:

$G = (V, E)$ (directed or undirected)

$U \subseteq V$, U is not empty

The subgraph of G induced by U is the subgraph with vertices in U and all edges (from G) of the following form

(a) (x, y), $x, y \in U$ (G directed), or

(b) $\{x, y\}$, $x, y \in U$ (G undirected)

This subgraph is denoted by <*U*>

A subgraph G' of a graph $G = (V, E)$ is called an *induced subgraph* if there exists $U \subseteq V$ such that G' = <*U*>

Note:

An induced subgraph is a subgraph

But a subgraph is not necessary to be an induced subgraph. Why?

<u>Def:</u>

$G = (V, E)$ (directed or undirected)

$G - v = (V - \{v\}, E')$, E' contains all edges of G except those that are incident with v

$G - e = (V, E - \{e\})$

<u>Def:</u>

$|V| = n$

K_n: (the complete graph on V) is a loop-free undirected graph where $\forall\ a, b \in V,\ a \neq b$, there is an edge $\{a, b\}$

(Hence, number of edges of $K_n = \dfrac{n^2 - n}{2}$)

<u>Examples:</u> $n = 4$, $n = 5$

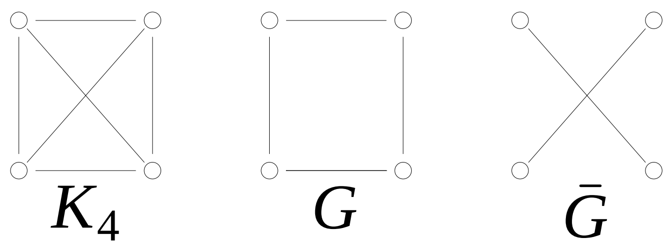
<u>Def:</u>

$G = (V, E)$: loop-free, undirected, $|V| = n$

$K_n = (V, E')$: complete graph on V

The *Complement* of G is defined as follows:

$$\bar{G} = (V, E' - E)$$



Def:

$G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$: undirected

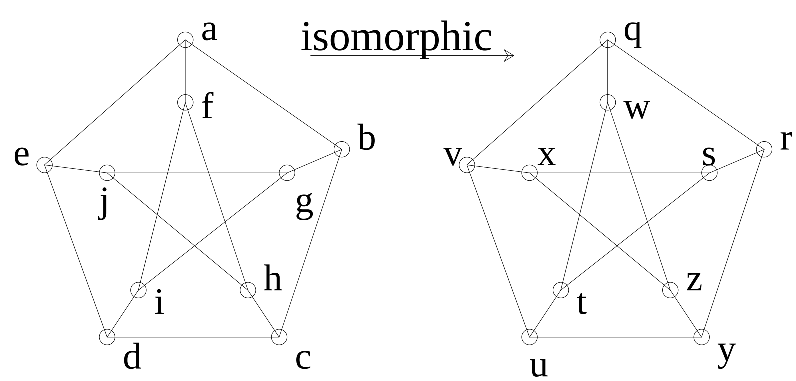
$f : V_1 \rightarrow V_2$ is a *graph isomorphism* if

(a) f is one-to-one and onto

(b) $\{a, b\} \in E_1$ iff $\{f(a), f(b)\} \in E_2$

In this case, G_1 and G_2 are called *isomorphic graphs* (i.e., G_1 and G_2 have the same structure).

Example:



$a \rightarrow q, b \rightarrow v, c \rightarrow u, d \rightarrow y, e \rightarrow r$
$f \rightarrow w, g \rightarrow x, h \rightarrow t, i \rightarrow z, j \rightarrow s$

Notes:

An isomorphism preserves adjacencies, hence, structures such as "paths" and "cycles".

How to determine if two graphs are isomorphic?

- If a cycle in G_1 does not have a counter part in G_2 then G_1 can not be isomorphic to G_2.
- If a path in G_1 does not have a counter part in G_2 then G_1 can not be isomorphic to G_2.
- Degrees of adjancy of corresponding vertices in isomorphic graphs must be the same.

Example:

The following two graphs are not isomorphic. Why?

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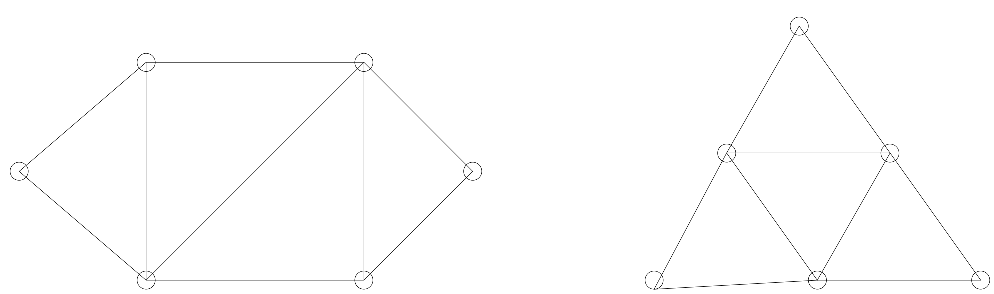

The number of vertices with degree of adjancy 2 is 2 in G_1 but the that number in G_2 is 3, or

The number of vertices with degree of adjancy 4 is 2 in G_1 but the that number in G_2 is 3, or

Each vertex of G_2 can be the start point of a trail which includes every edge of the graph. But in G_1, f and b are the only vertices with such a property.

Example:

If every induced subgraph of $G = (V, E)$, $|V| \geq 2$ is connected then G is isomorphic to K_n where $n = |V|$.

(Prove that

$$|E| = (n^2 - n)/2.$$

If $|E| < (n^2 - n)/2$ then it is possible to find an induced subgraph of G with two elements which is not connected.)

Example:

Find all (loop-free) non-isomorphic undirected graphs with four vertices. How many of them are connected?