3.1.1 Bezier Curve Segments of Degree 3

\[ C(t) = (1-t)^3 P_0 + 3t(1-t)^2 P_1 + 3t^2(1-t)P_2 + t^3P_3 \]

\[ 0 \leq t \leq 1 \]

Matrix form:

\[ C(t) = [1, t, t^2, t^3] \begin{bmatrix}
1 & 0 & 0 & 0 \\
-3 & 3 & 0 & 0 \\
3 & -6 & 3 & 0 \\
-1 & 3 & -3 & 1
\end{bmatrix} \begin{bmatrix}
P_0 \\
P_1 \\
P_2 \\
P_3
\end{bmatrix} = T \cdot M_b \cdot G \]
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- $P_i = (x_i, y_i)$ are called control points
- The polygon $P_0P_1P_2P_3$ is called the control polygon
- The weights $(1-t)^3$, $3t(1-t)^2$, $3t^2(1-t)$, and $t^3$ are called blending functions

Notes:
- Blending functions are always non-negative
- Blending functions always sum to 1
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- $C(0) = P_0; \quad C(1) = P_3$

  (A Bezier curve always starts at $P_0$ and ends at $P_3$.

- $C'(0) = 3(P_1 - P_0); \quad C'(1) = 3(P_3 - P_2)$

  (A Bezier curve is tangent to the control polygon at the endpoints)

- $C''(0) = 6(P_2 - 2P_1 + P_0); \quad C''(1) = 6(P_3 - 2P_2 + P_1)$

- Bezier curve segments satisfy convex hull property

  i.e., a Bezier curve segment is always contained in the convex hull of its control points
Bezier curves have intuitive appeal for interactive users.

3.1.2 General Bezier Curves

\[
C(t) = \sum_{i=0}^{n} B_{i,n}(t) P_i = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} P_i,
\]

where \( 0 \leq t \leq 1 \) and \( \binom{n}{i} = \frac{n!}{i! (n-i)!} \). \( B_{i,n}(t) \) are again called blending functions and \( P_i \) control points.

![Bezier Curve Diagram](image-url)
• All the properties mentioned on pages 54 and 55 hold for general Bezier curves.

A recurrence relation:

\[
C(t) = (1 - t) \left[ \sum_{i=0}^{n-1} B_{i,n-1}(t) P_i \right] + t \left[ \sum_{i=0}^{n-1} B_{i,n-1}(t) P_{i+1} \right] \\
= (1 - t) \cdot \left[ \sum_{i=0}^{n-1} \binom{n-1}{i} t^i (1 - t)^{n-1-i} P_i \right] + t \cdot \left[ \sum_{i=0}^{n-1} \binom{n-1}{i} t^i (1 - t)^{n-1-i} P_{i+1} \right]
\]
If degree = 3 then

\[
C(\frac{1}{3}) = \frac{2}{3} \left[ \frac{2}{3} \left( \frac{2}{3} P_0 + \frac{1}{3} P_1 \right) + \frac{1}{3} \left( \frac{2}{3} P_1 + \frac{1}{3} P_2 \right) \right]
+ \frac{1}{3} \left[ \frac{2}{3} \left( \frac{2}{3} P_1 + \frac{1}{3} P_2 \right) + \frac{1}{3} \left( \frac{2}{3} P_2 + \frac{1}{3} P_3 \right) \right]
\]
Midpoint Curve Subdivision

$P_0, M, N, O$ are control points of $C(t)$, $0 \leq t \leq 1/2$, and $O, P, Q, P_3$ are control points of $C(t)$, $1/2 \leq t \leq 1$.

Recursively subdivide the control polygons at the midpoints, we can divide the curve into many small segments, each with its own control points.

These control points, when connected, form a good linear approximation of the curve $C(t)$. (This linear approximation is usually used to find the intersection points of two Bezier curves)