14. Subdivision Surfaces

Becoming popular recently in graphical modeling, animation, and CAD/CAM because of

- *their stability in numerical computation*,

- *simplicity in coding*, and

- *their capability in modeling/representing complex shape of arbitrary topology*. 
14.1 B-Spline Curve/Surface Subdivision

14.1.1 Uniform Cubic B-spline Curve Segments

Given a uniform cubic B-spline curve segment defined by control points \( P_0, P_1, P_2 \) and \( P_3 \)

\[
C(t) = \frac{(1-t)^3}{6} P_0 + \frac{(4-6t^2+3t^3)}{6} P_1 + \frac{(1+3t+3t^2-3t^3)}{6} P_2 + \frac{t^3}{6} P_3
\]

\[0 \leq t \leq 1\]

How would you find the control points of the left half (or, right half) of \( C(t) \)?
Consider the matrix form of the curve

\[ \mathbf{C}(t) = [1, t, t^2, t^3] \frac{1}{6} \begin{bmatrix} 1 & 4 & 1 & 0 \\ -3 & 0 & 3 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} = T \cdot M \cdot G \]

If we define

\[ \bar{\mathbf{C}}(t) = \mathbf{C}\left(\frac{t}{2}\right), \quad 0 \leq t \leq 1 \]

(\( \bar{\mathbf{C}}(t) \) is the left half of \( \mathbf{C}(t) \))
we have

\[
\overline{C}(t) = [1, \frac{t}{2}, \frac{t^2}{4}, \frac{t^3}{8}] \cdot M \cdot G
\]

\[
= [1, t, t^2, t^3] \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1/2 & 0 & 0 \\
0 & 0 & 1/4 & 0 \\
0 & 0 & 0 & 1/8
\end{bmatrix} \cdot M \cdot G
\]

\[
= T \cdot S \cdot M \cdot G
\]

On the other hand, \( \overline{C}(t) \), as a uniform cubic B-spline curve segment, can also be expressed as

\[
\overline{C}(t) = T \cdot M \cdot G_1
\]

where \( G_1 \) is the control point polygon of \( \overline{C}(t) \)

\[
G_1^t = [\overline{P}_0, \overline{P}_1, \overline{P}_2, \overline{P}_3]
\]
Consequently, we have

\[ S \cdot M \cdot G = M \cdot G_1 \]

or

\[ G_1 = M^{-1} \cdot S \cdot M \cdot G = H \cdot G \]

where \( H = M^{-1} \cdot S \cdot M \). It is easy to see that

\[
M^{-1} = \begin{bmatrix}
1 & -1 & 2/3 & 0 \\
1 & 0 & -1/3 & 0 \\
1 & 1 & 2/3 & 0 \\
1 & 2 & 11/3 & 6
\end{bmatrix}
\]

Hence,

\[
H = \frac{1}{8} \begin{bmatrix}
4 & 4 & 0 & 0 \\
1 & 6 & 1 & 0 \\
0 & 4 & 4 & 0 \\
0 & 1 & 6 & 1
\end{bmatrix}
\]
and

\[
G_1 = \begin{bmatrix}
\frac{P_0 + P_1}{2} \\
\frac{P_0 + 6P_1 + P_2}{8} \\
\frac{P_1 + P_2}{2} \\
\frac{P_1 + 6P_2 + P_3}{8}
\end{bmatrix}
\]

What is the geometric meaning of this equation?
Similarly, one can show that $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, and $\mathbf{P}_4$ are the control points of the right half of $\mathbf{C}(t)$.

This shows that for a given control polygon with vertices $\mathbf{P}_i$, $i = 0, 1, \cdots n$, one can generate a uniform cubic B-spline curve by repeatedly cutting corners of the polygon in a two-stage process as follows:

0. Initially, set $\mathbf{P}_i^{(0)} = \mathbf{P}_i$ for $i = 0, 1, \cdots n$.

1. If $\mathbf{P}_i^{(k)}\mathbf{P}_{i+1}^{(k)}$ is a leg of the polygon after the $k$th corner-cutting iteration, generate two new points on the leg as follows:

$$Q_{2i} = \mathbf{P}_i^{(k)} + \frac{1}{4} (\mathbf{P}_{i+1}^{(k)} - \mathbf{P}_i^{(k)} )$$

$$Q_{2i+1} = \mathbf{P}_i^{(k)} + \frac{3}{4} (\mathbf{P}_{i+1}^{(k)} - \mathbf{P}_i^{(k)} )$$

2. Then generate vertices of the new polygon, $\mathbf{P}_i^{(k+1)}$, as follows:

$$\mathbf{P}_i^{(k+1)} = \frac{1}{2} (Q_i + Q_{i+1} )$$

where $i = 0, 1, \cdots, 2^{k+1}(n - 2) + 2$. 
14.1.2 Uniform Bicubic B-spline Surface Patches

Given a uniform bicubic B-spline surface patch defined by control points \( P_{i,j} \), \( 0 \leq i, j \leq 3 \),

\[
S(u,v) = U \cdot M \cdot G \cdot M^t \cdot V^t
\]

where \( M \) is defined in page 211 and

\[
U = [1, u, u^2, u^3], \quad V = [1, v, v^2, v^3], \quad G = \left[ P_{i,j} \right]_{3x3}
\]

How would you find the control points of the lower-left subpatch (\( 0 \leq u, v \leq 1/2 \)) of \( S(u,v) \)?
If we define

\[
\bar{S}(u,v) = S\left(\frac{u}{2}, \frac{v}{2}\right), \quad 0 \leq u, v \leq 1
\]

(\bar{S}(u,v) is the lower-left subpatch of \(S(u,v)\)), we have

\[
\bar{S}(u,v) = [1, \frac{u}{2}, \frac{u^2}{4}, \frac{u^3}{8}] \cdot M \cdot G \cdot M^t \cdot \begin{bmatrix} 1 \\ v/2 \\ v^2/4 \\ v^3/8 \end{bmatrix}
\]

\[
= U \cdot S \cdot M \cdot G \cdot M^t \cdot S^t \cdot V^t
\]

where

\[
S = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1/2 & 0 & 0 \\
0 & 0 & 1/4 & 0 \\
0 & 0 & 0 & 1/8
\end{bmatrix}
\]
On the other hand, $\bar{S}(u,v)$, as a uniform bicubic B-spline surface patch, can also be expressed as

$$\bar{S}(u,v) = U \cdot M \cdot G_1 \cdot M^t \cdot V^t$$

where $G_1$ is the control point matrix of $\bar{S}(u,v)$

$$G_1 = [P_{i,j}]_{3x3}$$

Consequently, we have

$$S \cdot M \cdot G \cdot M^t \cdot S^t = M \cdot G_1 \cdot M^t$$

or

$$G_1 = H \cdot G \cdot H^t$$

where

$$H = M^{-1} \cdot S \cdot M = \frac{1}{8} \begin{bmatrix} 4 & 4 & 0 & 0 \\ 1 & 6 & 1 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 1 & 6 & 1 \end{bmatrix}$$

(see page 213 for the computation of $H$)
Hence,

\[
\bar{P}_{0,0} = \frac{P_{0,0} + P_{1,0} + P_{0,1} + P_{1,1}}{4}
\]

\[
\bar{P}_{1,0} = \frac{1}{2} \left[ \frac{P_{0,0} + P_{2,0}}{2} + \frac{P_{1,0} + P_{1,1}}{2} \right]
\]

\[
\bar{P}_{1,1} = Q + \frac{R}{2} + \frac{P_{1,1}}{4}
\]

where

\[
Q = \frac{\bar{P}_{0,0} + \bar{P}_{2,0} + \bar{P}_{0,2} + \bar{P}_{2,2}}{4}
\]

\[
R = \frac{1}{4} \left[ \frac{P_{1,0} + P_{1,1}}{2} + \frac{P_{1,1} + P_{1,2}}{2} + \frac{P_{0,1} + P_{1,1}}{2} + \frac{P_{1,1} + P_{1,2}}{2} \right]
\]

\[
\ldots
\]
What is the geometric meaning of these equations?

**Face points:** $\vec{P}_{0,0}, \vec{P}_{2,0}, \vec{P}_{0,2}, \vec{P}_{2,2}, \ldots$

**Edge points:** $\vec{P}_{1,0}, \vec{P}_{3,0}, \vec{P}_{0,1}, \vec{P}_{0,3}, \ldots$

**Vertex points:** $\vec{P}_{1,1}, \vec{P}_{3,1}, \vec{P}_{1,3}, \vec{P}_{3,3}$

- Face point
- Edge point
- Vertex point
14.2 Catmull-Clark Subdivision Surfaces

How should we generalize the expressions on page 219 to a control point net (mesh) of arbitrary topology?

Rules for generating new vertices:

1. **New face points:** average of all the old points defining the face

2. **New edge points:** average of the midpoint of the old edge with the average of the two new face points of the faces sharing the edge

3. **New vertex points:** the average

\[
\frac{Q}{n} + \frac{2R}{n} + \frac{S(n-3)}{n}
\]

where

\(Q\) : average of the new face points of all faces adjacent to the old vertex point

\(R\) : average of the midpoints of all old edges incident to the old vertex point

\(S\) : old vertex point
Rules for generating new edges:

- Connecting each new face point to the new edge points of the edges defining the old face
- Connecting each new vertex point to the new edge points of all old edges incident to the old vertex point

New faces are then defined as those enclosed by new edges.

Example: generating a triangular patch to approximate A
Observation:
1. All new faces in (a) have four sides. Why?

2. Four new vertices in (a) have only 3 edges incident upon them. These points are called *extraordinary points*.

3. Six regions in (b) would generate bicubic B-spline patches.

4. All regions in (c) except 6 would generate bicubic B-spline patches.

5. In the limiting case, the entire triangular region, excluding the extraordinary points, is covered by a B-spline surface.

6. After one iteration the number of extraordinary points on the surface remains constant.

- What is the behavior of the limit surface at an extraordinary point?
14.3 Bi-quadratic Surfaces

The above method can also be applied to biquadratic B-splines.

The subdivision net is generated by creating a new face for each face, edge and vertex of the original net (this is done by creating a new vertex for each old vertex).

New control points:
For instance,

\[ P_{00} = \frac{9P_{00} + 3P_{01} + 3P_{10} + P_{11}}{16} \]

\[ = \frac{1}{4} \left[ \frac{P_{00} + P_{01} + P_{10} + P_{11}}{4} \right] + \frac{P_{00}}{4} \]

\[ + \frac{1}{2} \left[ \frac{P_{00} + P_{01}}{2} + \frac{P_{00} + P_{10}}{2} \right] \]

Hence, we have the following rule.

**Rule:** For each vertex \( P \) of each old face, generate a new vertex \( Q \) as follows:

\[ Q = \frac{F}{n} + \frac{2E}{n} + \frac{P(n-3)}{n} \]

where
\( n = \) number of vertices in the face

\[ F = \text{average of the vertices in the face} \]

\[ E = \text{average of the midpoints of the two edges incident on } P \]

Then connect each new vertex to the two adjacent new vertices in the same face and to the corresponding new vertices in adjacent faces

Example:
Question: how many extraordinary points the limit surface would have in this case?