Representation Theory for Default Logic

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Abstract

Default logic can be regarded as a mechanism to represent *families* of belief sets of a reasoning agent. As such, it is inherently second-order. In this paper, we study the problem of representability of a family of theories as the set of extensions of a default theory. We give a complete solution to the representability by means of normal default theories. We obtain partial results on representability by arbitrary default theories. In particular, we construct examples of denumerable families of non-including theories that are not representable. We also study the concept of equivalence between default theories. We show that for every normal default theory there exists a normal prerequisite-free theory with the same set of extensions. We derive a representation result connecting normal default logic with a version of *CWA*.

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1 Introduction

In this paper we investigate the issues related to the expressibility of default logic, a knowledge representation formalism introduced by Reiter [Rei80] and extensively investigated by the researchers of logical foundations of Artificial Intelligence [Eth88, Bes89, Bre91]. A default theory (D, W) describes a family (possibly empty) of belief sets of an agent reasoning with (D, W). In that, default logic is inherently second-order, but in a sense different from that used by logicians. Whereas a logical theory S describes a subset of the set of formulas (specifically, the set $Cn(S)$ of logical consequences of S), a default theory (D, W) describes a subset of the *powerset* of the set of all formulas, namely the family of all extensions of (D, W), $ext(D, W)$. Hence, default theories can be viewed as encodings of families of subsets of some universe described by a propositional language. For example, there are concise and intuitive encodings of a graph by means of default theories extensions of which describe all hamilton cycles, k-colorings, kernels and many other objects that can be associated with a graph [CMMT95]. This second-order flavor of default logic makes it especially useful in knowledge representation. An important question, then, is to characterize those families of sets that can be represented as the set of extensions of a certain default theory. This is the topic of our paper.

There is a constraint on the family $\mathcal{T}$ of extensions of a default theory (D, W). Namely such family must be *non-including* [Rei80]. In this paper we exhibit several classes of families of non-including theories that can be represented by default theories. We also show that there are non-representable families of non-including theories. The existential proof follows easily from a cardinality argument. There are continuum-many default theories in a given (denumerable) language, while there is more than continuum-many families of non-including theories. In the paper, we actually *construct* a family of non-including theories that is not represented by a default theory. Moreover, our family is denumerable (the cardinality argument mentioned above does not guarantee the existence of a non-representable denumerable family of non-including theories). On the other hand, we show that if one allows defaults with an infinite number of justifications, then every non-including family of theories is of the form $ext(D, W)$ for a suitably chosen *infinitary* theory (D, W).

The family of extensions of a *normal* default theory is not only non-including, but all its members are pairwise inconsistent [Rei80]. In this paper, we fully characterize these families of theories which are of the form $ext(D, W)$, for a normal default theory (D, W). In addition, we construct examples of denumerable families of pairwise inconsistent theories which are not representable by normal default theories.

Given a family of theories $\mathcal{T}$, a default theory (D, W) such that $ext(D, W) = \mathcal{T}$ is not unique. Thus, it is natural to search for simpler default theories (D', W') with the same set of extensions as (D, W). Let us call (D', W') equivalent to (D, W) if $ext(D, W) = ext(D', W')$. We show that for every (D, W) we can effectively (without constructing extensions of (D, W)) find an equivalent theory (D', W') with all defaults in D' prerequisite-free (this result was obtained independently by Schaub [Sch92], and Bonatti and Eiter [BE95]). An important feature of our approach is that it shows that when (D, W) is normal, we can construct a *normal* prerequisite-free default theory (D', W') equivalent to (D, W).

We also present results that allow us to replace some normal theories (D, W) with equivalent normal default theories of the form $(D', \emptyset)$. At present, it is an open problem to decide whether such replacement is possible for every normal default theory (D, W) with W consistent.

We discuss yet another (weaker) form of equivalence and prove that every normal default theory is equivalent to a theory closely related to the closed world assumption over a certain set of atoms.

This paper sheds some light on the issue of expressibility of default logic and, in particular, on expressibility of normal default logic. We firmly believe that the success of default logic as a knowledge representation mechanism depends on a deeper understanding of expressibility issues.

2 Preliminaries

In this paper, by $\mathcal{L}$ we denote a language of propositional logic with a denumerable set of atoms At. By a *theory* we always mean a subset of $\mathcal{L}$ *closed under propositional provability.* Let B be a set of standard monotone inference rules. The formal system obtained by extending propositional calculus with the rules from B will be denoted by $PC + B$. The corresponding provability operator will be denoted by $\vdash_B$ and the consequence operator by

$Cn^B(\cdot)$ [MT93].

A *default* is an expression d of the form $\frac{\alpha:\Gamma}{\beta}$, where α and β are formulas from $\mathcal{L}$ and Γ is a **finite** subset of $\mathcal{L}$. The formula α is called the *prerequisite*, formulas in Γ — the *justifications*, and β — the *consequent* of d. The prerequisite, the set of justifications and the consequent of a default d are denoted by $p(d)$, $j(d)$ and $c(d)$, respectively. If $p(d)$ is a tautology, d is called *prerequisite-free* ($p(d)$ is then usually omitted from the notation of d). This terminology is naturally extended to a set of defaults D.

A pair (D, W), where D is a set of defaults and W is a set of formulas, is called a *default theory*. A default theory (D, W) is called *finite* if both D and W are *finite*. For a set of defaults D, define

$$Mon(D) = \left\{ \frac{p(d)}{c(d)} : d \in D \right\}.$$

A default d (a set of defaults D) is *applicable* with respect to a theory S (is S-*applicable*) if $S \not\vdash \neg\gamma$ for every $\gamma \in j(d)$ ($j(D)$, respectively). Let D be a set of defaults. By the *reduct* D_S of D with respect to S we mean the set of monotone inference rules:

$$D_S = Mon(\{d \in D : d \text{ is } S\text{-applicable}\}).$$

A theory S is an *extension*[4] of a default theory (D, W) if and only if

$$S = Cn^{D_S}(W).$$

The family of all extensions of (D, W) is denoted by $ext(D, W)$.

Let S be a theory. A default d is *generating* for S if d is S-applicable and $p(d) \in S$. The set of all defaults in D generating for S is denoted by $GD(D, S)$. It is well-known [MT93] that

(P1) If S is an extension of (D, W) then $S = Cn(W \cup c(GD(D, S)))$,

(P2) If all defaults in D are prerequisite-free then S is an extension of (D, W) if and only if $S = Cn(W \cup c(GD(D, S)))$.

We will define now the key concepts of the paper.

[4]Our definition of extension is different from but equivalent to the original definition by Reiter. See [MT93] for details.

Definition 2.1 Default theories Δ and Δ' are *equivalent* if $ext(\Delta) = ext(\Delta')$.

Definition 2.2 Let Δ be a default theory over a language $\mathcal{L}$ and let Δ' be a default theory over a language $\mathcal{L}'$ such that $\mathcal{L} \subseteq \mathcal{L}'$. Theory Δ is semi-equivalent to Δ' if $ext(\Delta) = \{T \cap \mathcal{L} : T \in ext(\Delta')\}$.

Definition 2.3 Let $\mathcal{T}$ be a family of theories contained in $\mathcal{L}$. The family $\mathcal{T}$ is *representable* by a default theory Δ if $ext(\Delta) = \mathcal{T}$.

In the paper we will sometimes allow the set of justifications in defaults to be infinite. In this case we will talk about *infinitary* defaults and default theories. All definitions in this section extend naturally to the case of infinitary default theories. In particular, Properties (P1) and (P2) hold for infinitary default theories, too.

3 Default theories without normality restriction

We start with the result that allows us to replace any default theory with an equivalent default theory in which all defaults are prerequisite-free. As mentioned, this result was known before. However, our argument shows that if we start with a normal default theory, its prerequisite-free equivalent replacement can also be chosen to be normal.

Theorem 3.1 *For every default theory Δ there is a prerequisite-free default theory Δ' equivalent to Δ. Moreover, if Δ is normal then Δ' can be chosen to be normal, too.*

Proof: Let $\Delta = (D, W)$. By a *quasi-proof* from D and W we mean any proof from W in the system $PC + Mon(D)$. For every quasi-proof ϵ from D and W let D_ϵ be the set of all defaults used in ϵ. For each such proof ϵ, define

$$d_\epsilon = \frac{: j(D_\epsilon)}{\bigwedge cons(D_\epsilon)}.$$

Next, define

$$E = \{d_\epsilon : \epsilon \text{ is a quasi-proof from } W\}.$$

Each default in E is prerequisite-free. Put $\Delta' = (E, W)$. We will show that Δ' has exactly the same extensions as (D, W). To this end, we will show that for every theory S and for every formula φ,

$$W \vdash_{D_S} \varphi \ \text{ iff } \ W \vdash_{E_S} \varphi.$$

Assume first that $W \vdash_{D_S} \varphi$. Then, there is a quasi-proof ϵ of φ such that all defaults in D_ϵ are applicable with respect to S. Moreover, $W \cup c(D_\epsilon) \vdash \varphi$. Observe that $c(d_\epsilon) \vdash c(D_\epsilon)$. Since d_ϵ is prerequisite-free and S-applicable, $W \vdash_{E_S} W \cup c(D_\epsilon)$. Hence, $W \vdash_{E_S} \varphi$.

To prove the converse implication, observe that since all defaults in E are prerequisite-free,

$$\{\varphi : W \vdash_{E_S} \varphi\} = Cn(W \cup c(E_S)).$$

Hence, it is enough to show that

$$W \vdash_{D_S} W \cup c(E_S).$$

Clearly, for every $\varphi \in W$, $W \vdash_{D_S} \varphi$. Consider then $\varphi \in c(E_S)$. It follows that there is a quasi-proof ϵ such that d_ϵ is S-applicable and $c(d_\epsilon) = \varphi$. Consequently, all defaults occurring in ϵ are S-applicable. Thus, for every default $d \in D_\epsilon$,

$$W \vdash_{D_S} c(d).$$

Since $\varphi = \bigwedge c(D_\epsilon)$,

$$W \vdash_{D_S} \varphi.$$

To prove the claim for normal default theories, observe that if each default in D is normal, then each default in E is of the form

$$\frac{:\Gamma}{\bigwedge \Gamma}.$$

Let $\widehat{E}$ be a set of defaults obtained from E by replacing each default $\frac{:\Gamma}{\bigwedge\Gamma}$ by the normal default $\frac{:\bigwedge\Gamma}{\bigwedge\Gamma}$. It is easy to see that S is an extension of (E, W) if and only if S is an extension of $(\widehat{E}, W)$. $\square$

Let us comment here that the first part of the assertion, dealing with arbitrary default theories, holds also for infinitary default theories (essentially the same argument works). The second part, concerned with normal default theories, can be extended to the infinitary case under a suitable generalization of normal default logic to the infinitary case.

The next result fully characterizes families of theories representable by default theories with a finite set of defaults.

Theorem 3.2 *The following statements are equivalent:*

(i) $\mathcal{T}$ *is representable by a default theory* (D, W) *with finite* D

(ii) $\mathcal{T}$ *is a finite set of non-including theories, finitely generated over the intersection of* $\mathcal{T}$

Proof: Assume (i). Since every extension of (D, W) is of the form $Cn(W \cup c(D'))$, for some $D' \subseteq D$, it follows that $ext(D, W)$ is finite. It is also well-known ([Rei80, MT93]) that $ext(D, W)$ are non-including. Let U be the intersection of all theories in $ext(D, W)$. Then $W \subseteq U$. Consequently, each extension in $ext(D, W)$ is of the form $Cn(U \cup c(D'))$. Hence, each extension is finitely generated over the intersection of $ext(D, W)$.

Now, assume (ii). Let U be the intersection of all theories in $\mathcal{T}$. It follows that there is a positive integer k and formulas $\varphi_1, \ldots, \varphi_k$ such that $\mathcal{T} = \{T_1, \ldots, T_k\}$ and each $T_i = Cn(U \cup \{\varphi_i\})$.

Assume first that $k = 1$. Then, it is evident that $\mathcal{T}$ is the family of extensions of the default theory $(\emptyset, T_1)$. Hence, assume that $k \geq 2$. Since the theories in $\mathcal{T}$ are non-including, for every $j \neq i$ we have

$$U \cup \{\varphi_i\} \not\vdash \varphi_j. \tag{1}$$

In particular, each theory in $\mathcal{T}$ is consistent and so is U. Moreover, it follows from (1) that for every $j = 1, \ldots, k$,

$$U \not\vdash \varphi_j. \tag{2}$$

Define

$$d_i = \frac{:\{\neg\varphi_1, \ldots, \neg\varphi_{i-1}, \neg\varphi_{i+1}, \ldots, \neg\varphi_k\}}{\varphi_i},$$

$i = 1, \ldots, k$. Next, define $D = \{d_1, \ldots, d_k\}$. We will show that $ext(D, U) = \mathcal{T}$.

Let T be an extension of (D, U). Then, there is a subset Φ of $\{\varphi_1, \ldots, \varphi_k\}$ such that $T = Cn(U \cup \Phi)$. Assume that $|\Phi| = 0$. Then, by (2), $D_T = \{\frac{:}{\varphi_i} : i = 1, \ldots, k\}$. Consequently, $U = T = Cn^{D_T}(U) = Cn(U \cup \{\varphi_1, \ldots, \varphi_k\})$. Hence, for every i, $U \vdash \varphi_i$, a contradiction (with (2)). Hence, $|\Phi| > 0$. Assume that $|\Phi| > 1$. By the definition of D, $D_T = \emptyset$. Consequently, $T = Cn(U \cup \Phi) = Cn^{D_T}(U) = Cn(U)$. Let $\varphi \in \Phi$ (recall that $\Phi \neq \emptyset$). Then, $U \vdash \varphi$, a contradiction. Hence, every extension T of (D, W) is of the form $Cn(U \cup \{\varphi_i\})$ for some i, $1 \leq i \leq k$.

To complete the proof, consider an arbitrary i, $1 \leq i \leq k$. We will show that T_i is an extension of (D, W). First, observe that, by (1), $D_{T_i} = \{\frac{:}{\varphi_i}\}$. Consequently, $Cn^{D_{T_i}}(U) = Cn(U \cup \{\varphi_i\}) = T_i$. Hence, T_i is an extension of (D, U). $\square$

This result and its argument provide the following corollary which gives a complete characterization of families of theories representable by finite default theories, that is, theories (D, W) with both D and W finite.

Corollary 3.3 *The following statements are equivalent:*

1. *$\mathcal{T}$ is representable by a finite default theory*
2. *$\mathcal{T}$ is a finite set of finitely generated non-including theories*

As pointed out in the introduction, the cardinality argument implies the existence of non-representable families of non-including theories. However, it does not imply the existence of denumerable non-representable families. We will now show two examples of such families. The first family consists of non-including finitely generated theories. The second one consists of mutually inconsistent theories.

Theorem 3.4 *There exists a countable family of finitely generated non-including theories $\mathcal{T}$ such that $\mathcal{T}$ is not representable by a finitary default theory.*

Proof: Let $\{p_0, p_1, \ldots\}$ be a set of propositional atoms. Define $T_i = Cn(\{p_i\})$, $i = 0, 1, \ldots$, and $\mathcal{T} = \{T_i : i = 0, 1, \ldots\}$. It is clear that $\mathcal{T}$ is countable and consists of non-including theories. We will show that $\mathcal{T}$ is not representable by a default theory.

Assume that $\mathcal{T}$ is represented by a default theory (D, W). By Theorem 3.1, we may assume that all defaults in D are prerequisite-free. We can also assume that no default in D contains a justification which is contradictory (such defaults are never used to construct extensions).

Consider a default $d \in D$. Since $j(d)$ is finite, there is k such that for all $m > k$, all formulas in $j(d)$ are consistent with T_m. Since T_m is an extension of (D, W), $c(d) \in T_m$, for $m > k$. Since

$$\bigcap_{m>k} T_m = Cn(\emptyset),$$

$c(d)$ is a tautology. Since d was arbitrary, it follows that (D, W) possesses only one extension, namely $Cn(W)$, a contradiction. $\Box$

Theorem 3.5 *There exists a countable family of mutually inconsistent theories $\mathcal{T}$ such that $\mathcal{T}$ is not representable by a default theory. In particular $\mathcal{T}$ is not representable by a normal default theory.*

Proof: Let $\{p_0, p_1, \ldots\}$ be a set of propositional atoms. Define

$$T_i = Cn(\{\neg p_i, p_{i+1}, \ldots\}),$$

for $i = 0, 1, \ldots$, and $\mathcal{T} = \{T_i : i = 0, 1, \ldots\}$. It is clear that $\mathcal{T}$ is countable and consists of pairwise inconsistent theories. Now, we apply precisely the same argument as in the proof of Theorem 3.4. $\Box$

Our counterexamples have an additional property that their infinite subsets and all supersets are also counterexamples.

Finally, we show that if infinite sets of justifications are allowed in defaults, every theory of non-including theories can be represented as the family of extensions.

Theorem 3.6 *The following statements are equivalent for a set of theories $\mathcal{T}$:*

1. *$\mathcal{T}$ is representable by an infinitary default theory Δ*

2. *$\mathcal{T}$ is a set of non-including theories.*

When $\mathcal{T}$ is finite, Δ can be chosen finitary.

Proof: Let $\mathcal{T}$ be the set of extensions of an infinitary default theory (D, W). Assume that $S, T \in \mathcal{T}$ and $S \subseteq T$. Then, $D_T \subseteq D_S$. Hence,

$$T = Cn^{D_T}(W) \subseteq Cn^{D_S}(W) = S$$

Consequently, $T = S$. Hence, $\mathcal{T}$ consists of non-including theories.

Conversely, let $\mathcal{T}$ consist of non-including theories. If $\mathcal{T} = \{T\}$, then define $D = \emptyset$. Clearly, $ext(D, T) = \mathcal{T}$.

Hence, assume that $\mathcal{T}$ contains at least two theories. Since they are non-including, it follows that all theories contained in $\mathcal{T}$ are consistent.

For every $S, T \in \mathcal{T}$ such that $S \neq T$, define $\varphi_{S,T}$ to be any formula belonging to $S \setminus T$. For every $T \in \mathcal{T}$, define

$$D^T = \left\{ \frac{: \{\neg\varphi_{S,T} \colon S \in \mathcal{T},\ S \neq T\}}{\varphi} \colon \varphi \in T \right\}.$$

Finally, define

$$D = \bigcup_{T \in \mathcal{T}} D^T.$$

We will show that $ext(D, \emptyset) = \mathcal{T}$.

Consider $T \in \mathcal{T}$. Then $D_T = \{\frac{:}{\varphi} \colon \varphi \in T\}$. Hence, $Cn^{D_T}(\emptyset) = T$ and T is an extension of (D, W).

Conversely, let T be an extension of (D, W). We have just proved that $\mathcal{T} \subseteq ext(D, W)$. Consequently, (D, W) has at least two extensions. It follows that $Cn(\emptyset)$ is not an extension of (D, W). In particular, $T \neq Cn(\emptyset)$. Consequently, the set D_T is not empty.

Consider a set $S \in \mathcal{T}$. Observe that all defaults in D^S have the same set of justifications. Consequently, either all of them are generating for T or none. It follows that T is the union of a nonempty (since $D_T \neq \emptyset$) family of theories in $\mathcal{T}$. If T is the union of at least two theories, than $D_T = \emptyset$, a contradiction. Hence, $T = S$, for some $S \in \mathcal{T}$. That is, $T \in \mathcal{T}$. □

4 Pruning the set of extensions

For every default

$$d = \frac{\alpha : M\beta_1, \ldots, M\beta_k}{\gamma}$$

define

$$d' = \frac{\alpha : M\beta_1, \ldots, M\beta_k, M\top}{\gamma}.$$

For a set of defaults D, define

$$D' = \{d' \colon d \in D\}.$$

Let $\varphi \in \mathcal{L}$. Define

$$d_\varphi = \frac{\varphi :}{\bot}.$$

Theorem 4.1 *Let $E \subseteq \mathcal{L}$ be consistent and let (D, W) be a default theory. Then, E is an extension of $(D' \cup \{d_\varphi\}, W)$ if and only if $\varphi \notin E$ and E is an extension of (D, W).*

Proof: Since E is consistent,

$$(D' \cup \{d_\varphi\})_E = D_E \cup \{\frac{\varphi :}{\bot}\}.$$

Assume that $\varphi \notin E$ and that E is an extension of (D, W). Then

$$E = Cn^{D_E}(W)$$

and $\varphi \notin Cn^{D_E}(W)$. Consequently,

$$E = Cn^{D_E}(W) = Cn^{D_E \cup \{\frac{\varphi}{\bot}\}}(W) = Cn^{(D' \cup \{d_\varphi\})_E}(W).$$

Hence, E is an extension of $(D' \cup \{d_\varphi\}, W)$.

Conversely, assume that E is an extension of $(D' \cup \{d_\varphi\}, W)$. Then,

$$E = Cn^{(D' \cup \{d_\varphi\})_E}(W) = Cn^{D_E \cup \{\frac{\varphi}{\bot}\}}(W).$$

Since E is consistent, it follows that $\varphi \notin Cn^{D_E}(W)$. Consequently,

$$Cn^{D_E}(W) = Cn^{D_E \cup \{\frac{\varphi}{\bot}\}}(W) = E.$$

Hence, $\varphi \notin E$ and E is an extension of (D, W). $\square$

By a *pc-theory* we mean a theory that is closed under propositional consequence. We say that a family $\mathcal{F}$ of pc-theories has a *strong system of distinct representatives* (SSDR, for short) if for every $F \in \mathcal{F}$ there is a formula $\varphi_F \in F$ which does not belong to any other theory in $\mathcal{F}$.

Theorem 4.2 *If $\mathcal{F}$ is representable by a default theory and has an SSDR, then every family $\mathcal{G} \subseteq \mathcal{F}$ is representable by a default theory.*

Proof: The claim is obvious if $\mathcal{F} = \mathcal{L}$. So, assume that all members of $\mathcal{F}$ are consistent (since $\mathcal{F}$ is an antichain, there are no other possibilities). Let (D, W) be a default theory such that $Ext(D, W) = \mathcal{F}$. Define

$$\overline{D} = D' \cup \{d_{\varphi_F} \colon F \in \mathcal{F} \setminus \mathcal{G}\}.$$

Since all theories in $\mathcal{F}$ are consistent, the assertion follows from the definition of an SSDR and from Theorem 4.1. □

Remarks: There are families of pc-theories which possess SSDRs but which are not representable by a default theory (cf. examples in the paper). Not every subfamily of a representable family is representable. It follows by the cardinality argument from the the fact that there are representable families of cardinality continuum.

5 Normal default theories

Our first result in this section describes the family of extensions of an arbitrary prerequisite-free normal default theory.

Theorem 5.1 *Let $W, \Psi \subseteq \mathcal{L}$. Let $D = \{\frac{:\varphi}{\varphi} \colon \varphi \in \Psi\}$. If W is inconsistent then $ext(D, W) = \{\mathcal{L}\}$. Otherwise, $ext(D, W)$ is exactly the family of all theories of the form $Cn(W \cup \Phi)$, where Φ is a maximal subset of Ψ such that $W \cup \Phi$ is consistent.*

Proof: The case of inconsistent W is evident. Hence, let us assume that W is consistent. Let T be an extension of (D, W). Since W is consistent, T is consistent, too. Let $\Phi = \{\varphi \in \Psi \colon T \not\vdash \neg\varphi\}$. Clearly, $\Phi = c(GD(D, T))$. By (P2), $T = Cn(W \cup \Phi)$. Moreover, since T is consistent, $W \cup \Phi$ is consistent. We will show that Φ is a maximal subset of Ψ with this property. Let Φ' be such that $\Phi \subseteq \Phi' \subseteq \Psi$. Assume that $W \cup \Phi'$ is consistent. Then, $T \cup \Phi'$ is consistent. Hence, $\Phi' \subseteq \Phi$ and, consequently, $\Phi = \Phi'$.

Assume next that $T = Cn(W \cup \Phi)$, where Φ is a maximal subset of Ψ such that $W \cup \Phi$ is consistent. Then, it is easy to see that

$$GD(D, T) = \left\{\frac{:\varphi}{\varphi} \colon \varphi \in \Phi\right\}.$$

Hence, $\Phi = c(GD(D,T))$ and $T = Cn(W \cup c(GD(D,T)))$. Since all defaults in D are prerequisite-free, it follows by the property (P2) that T is an extension of (D,W). $\Box$

As a corollary, we obtain a full characterization of families of theories that are representable by normal default theories.

Corollary 5.2 *A family $\mathcal{T}$ of theories in $\mathcal{L}$ is representable by a normal default theory if and only if $\mathcal{T} = \{\mathcal{L}\}$ or there is a consistent set of formulas W and a set of formulas Ψ such that $\mathcal{T} = \{Cn(W \cup \Phi) : \Phi \subseteq \Psi$ and Φ is maximal so that $W \cup \Phi$ is consistent$\}$.*

Proof: By Theorem 3.1, $\mathcal{T}$ is representable by a normal default theory if and only if it is representable by a normal default theory with all defaults prerequisite-free. Hence, the assertion follows from Theorem 5.1. $\Box$

Corollary 5.3 *A family of theories $\mathcal{T}$ is representable by a normal default theory with empty objective part if and only if there is a set of formulas Ψ such that $\mathcal{T} = \{Cn(\Phi) : \Phi$ is maximal consistent subset of $\Psi\}$.* $\Box$

Next, we explore the connections of normal default logic with the Closed World Assumption. Consider a set of atoms P. Define the set of defaults

$$D^{CWA^P} = \left\{ \frac{:\neg p}{\neg p} : p \in P \right\}.$$

Informally, a default $\frac{:\neg p}{\neg p}$ allows us to derive $\neg p$ if p is not derivable. This has the flavor of the Closed World Assumption. The exact connection with *CWA* is given by the following result [MT93]: If $P = At$ then W is *CWA*-consistent if and only if (D^{CWA^P}, W) possesses a unique consistent extension.

Theorem 5.4 *For every normal default theory (D,W) in $\mathcal{L}$ there exists a language $\mathcal{L}' \supseteq \mathcal{L}$, a set of atoms P in $\mathcal{L}'$, and $W' \subseteq \mathcal{L}'$ such that (D,W) is semi-equivalent to a default theory (D^{CWA^P}, W').*

Proof: By Theorem 3.1 we can assume that all defaults in D are prerequisite-free. Let Ψ be the set of consequents of defaults in D. For each $\psi \in \Psi$ select a new atom not belonging to At (recall that At is the set of atoms in $\mathcal{L}$). This atom is denoted by p_ψ and the set P is defined as $\{p_\psi : \psi \in \Psi\}$. Define now $\mathcal{L}'$ to be the language generated by the set of atoms $At' = At \cup P$. Next, define V as this set of formulas:

$$\{\neg p_\psi \Leftrightarrow \psi \colon \psi \in \Psi\}.$$

We notice the following fact:

(F1) Let $\Phi \subseteq \Psi$. Then $W \cup \Phi$ is consistent if and only if $W \cup V \cup \{\neg p_\psi : \psi \in \Phi\}$ is consistent.

Indeed, for a model v of $W \cup \Phi$, define v' as follows:

$$v'(p) = \begin{cases} v(p) & \text{if } p \in At \\ 1 - v(\psi) & \text{if } p = p_\psi \end{cases}$$

It is clear that v' is a model of $W \cup V \cup \{\neg p_\psi : \psi \in \Phi\}$. Conversely, when v', a valuation of At' is a model of $W \cup V \cup \{\neg p_\psi : \psi \in \Phi\}$ then $v = v'|_{At}$ is a model of $W \cup \{\psi : \psi \in \Phi\}$. Hence, (F1) follows.

Observation (F1) implies that Φ is a maximal subset of Ψ consistent with W if and only if $\{\neg p_\psi : \psi \in \Phi\}$ is a maximal subset of $\{\neg p_\psi : \psi \in \Phi\}$ which is consistent with $W \cup V$.

Next, observe that if Φ is a maximal set of formulas contained in Ψ and consistent with W then for all $\theta \in \Psi \setminus \Phi$

$$W \cup V \cup \{\neg p_\psi : \psi \in \Phi\} \vdash p_\theta.$$

We are now ready to construct the desired default theory. We put $W' = W \cup V$ and $D' = \{\frac{:\neg p_\psi}{\neg p_\psi} : \psi \in \Psi\}$. Clearly, $D' = CWA^P$. Using the observations listed above, it is easy to show that the theory (D', W') semi-represents $ext(D, W)$. The details will be given in the full version of the paper. $\Box$

We will next study the issue of equivalence between normal default theories. We have already seen that we can replace any normal default theory with an equivalent normal prerequisite-free one (Theorem 3.1). The problem of interest now will be to establish when a normal default theory can be replaced by an equivalent normal default theory with empty objective part. We have only partial answers to this problem.

First, consider a normal default theory (D, W) such that W is inconsistent. Then $ext(D, W) = \{\mathcal{L}\}$. On the other hand, for every set of normal defaults D', $ext(D', \emptyset)$ contains only consistent extensions. Hence, (D, W) is not equivalent to any normal default theory with empty objective part. From now on we will focus on normal default theories (D, W) for which W is consistent.

Theorem 5.5 *For every normal default theory (D, W) with W consistent and finite there exists a prerequisite-free normal default theory $(D', \emptyset)$ equivalent to (D, W).*

Proof: By Theorem 3.1, without loss of generality we can assume that each default in D is prerequisite-free. Define $\omega = \bigwedge W$.

First, assume that the justification of every default in D is inconsistent with ω. Then, $ext(D, W) = \{Cn(W)\}$. Let $D' = \{\frac{:\omega}{\omega}\}$. Clearly, $ext(D', \emptyset) = ext(D, W)$.

Hence, assume that there are defaults in D whose justifications are consistent with ω. For every default $d = \frac{:\beta}{\beta}$ in D, define $d_\omega = \frac{:\beta\wedge\omega}{\beta\wedge\omega}$. Finally, define $D' = \{d_\omega : d \in D\}$. The statement now follows from Theorem 5.1 $\Box$.

Our next two results are concerned with normal default theories of a special form. Let us denote by D^{COMP} the following set of defaults:

$$\left\{\frac{:q}{q} : q \text{ is a literal in } \mathcal{L}\right\}.$$

Proposition 5.6 *For every consistent theory W the extensions of (D^{COMP}, W) are precisely the theories T such that T is complete, consistent and $W \subseteq T$.*

Proof: If T is an extension of (D^{COMP}, W) then $W \subseteq T$, T is complete (because all the atoms of $\mathcal{L}$ are conclusions of defaults in D) and consistent (because W is consistent).

Conversely, let T be a complete, consistent theory such that $W \subseteq T$. Let T_L be the set of all literals in T. It is clear that $D_T = \{\frac{:}{q} : q \in T_L\}$. Hence, $T = Cn^{D_T}(W)$. Consequently, T is an extension of (D, W). $\Box$

Proposition 5.7 *For every consistent W, the family $\mathcal{T}_W = \{T : T$ is complete, consistent, and $W \subseteq T\}$ is representable by a normal prerequisite-free default theory with empty-objective part.*

Proof: Proposition 5.6 implies that we can represent $\mathcal{T}_W$ by a default theory (D^{COMP}, W). We will now construct another set of defaults D such that $(D, \emptyset)$ represents $\mathcal{T}_W$.

If the language $\mathcal{L}$ has only a finite number of atoms, we can assume that W is finite. Hence, the assertion follows from Theorem 5.5. So assume that $\mathcal{L}$ has infinite number of atoms. Let $p_0, p_1, \ldots$ be an enumeration of atoms in $\mathcal{L}$. For the purpose of this proof we define, for an atom p, $0p = p$, $1p = \neg p$.

Consider a set $tree^W$ of all finite sequences $\langle \epsilon_0 p_0, \ldots, \epsilon_n p_n \rangle$ such that $W \cup \{\epsilon_0 p_0, \ldots, \epsilon_n p_n\}$ is consistent.

The set $tree^W$ forms a *tree*. That is, if $\langle \epsilon_0 p_0, \ldots, \epsilon_n p_n \rangle$ belongs to $tree^W$ and $m < n$ then also $\langle \epsilon_0 p_0, \ldots, \epsilon_m p_m \rangle$ belongs to $tree^W$.

An infinite branch in $tree^W$ determines the infinite sequence $\langle \epsilon_0 p_0, \ldots \rangle$. Since for all n, $W \cup \{\epsilon_0 p_0, \ldots, \epsilon_n p_n\}$ is consistent, $Cn(\{\epsilon_0 p_0, \ldots, \epsilon_n p_n, \ldots\})$ is also consistent. It is also complete and therefore, as W is consistent with it, $W \subseteq Cn(\{\epsilon_0 p_0, \ldots\})$. Conversely, if T is consistent and complete then there is a sequence $\langle \epsilon_0 p_0, \ldots \rangle$ such that $T = Cn(\{\epsilon_0 p_0, \ldots\})$. If $W \subseteq T$ then for all n, $W \cup \{\epsilon_0 p_0, \ldots, \epsilon_n p_n\}$ is consistent. Thus we proved that there is a one-to-one correspondence between the branches in $tree^W$ and complete, consistent theories containing W.

Now define:

$$D = \left\{ \frac{: \epsilon_0 p_0 \wedge \ldots \wedge \epsilon_n p_n}{\epsilon_0 p_0 \wedge \ldots \wedge \epsilon_n p_n} : \langle \epsilon_0 p_0, \ldots, \epsilon_n p_n \rangle \in tree^W \right\}$$

We show that the extensions of $(D, \emptyset)$ are precisely the theories of the form $Cn(\{\epsilon_0 p_0, \ldots, \epsilon_n p_n, \ldots\})$, where $\langle \epsilon_0 p_0, \ldots, \epsilon_n p_n, \ldots \rangle$ is an infinite branch through $tree^W$.

Indeed, if $\langle \epsilon_0 p_0, \ldots, \epsilon_n p_n, \ldots \rangle$ is an infinite branch through $tree^W$ then $T = Cn(\{\epsilon_0 p_0, \ldots, \epsilon_n p_n, \ldots\})$ is a complete theory. The only default rules in D that have conclusions in T are the rules $\frac{:\epsilon_0 p_0 \wedge \ldots \wedge \epsilon_n p_n}{\epsilon_0 p_0 \wedge \ldots \wedge \epsilon_n p_n}$ for $n \in N$. This implies that $Cn(\{\epsilon_0 p_0, \ldots, \epsilon_n p_n, \ldots\})$ is an extension of $(D, \emptyset)$.

Conversely, if T is an extension of $(D, \emptyset)$ then if $\frac{:\epsilon_0 p_0 \wedge \ldots \wedge \epsilon_n p_n}{\epsilon_0 p_0 \wedge \ldots \wedge \epsilon_n p_n}$ is a generating rule for T then for all $m < n$, $\frac{:\epsilon_0 p_0 \wedge \ldots \wedge \epsilon_n p_m}{\epsilon_0 p_0 \wedge \ldots \wedge \epsilon_n p_m}$ is also a generating rule for T. Next, since $W = \emptyset$, T must be consistent. This means that if two rules $\frac{:\epsilon_0 p_0 \wedge \ldots \wedge \epsilon_n p_n}{\epsilon_0 p_0 \wedge \ldots \wedge \epsilon_n p_n}$ and $\frac{:\delta_0 p_0 \wedge \ldots \wedge \delta_m p_m}{\delta_0 p_0 \wedge \ldots \wedge \delta_m p_m}$ are both generating for T then $m \leq n$ and $\delta_0 = \epsilon_0, \ldots \delta_m = \epsilon_m$ or $n \leq m$ and $\delta_0 = \epsilon_0, \ldots \delta_n = \epsilon_n$. Thus, in order to complete our argument it is enough to show that the set of generating rules for T is infinite. Assume otherwise. Then, there exists a sequence $\langle \epsilon_0 p_0, \ldots, \epsilon_n p_n \rangle$ such that $T = Cn(\{\epsilon_0 p_0, \ldots, \epsilon_n p_n\})$. But recall that $\langle \epsilon_0 p_0, \ldots, \epsilon_n p_n \rangle \in \mathit{tree}^W$. Therefore $W \cup \{\epsilon_0 p_0, \ldots, \epsilon_n p_n\}$ is consistent and so there is a complete theory T' containing $W \cup \{\epsilon_0 p_0, \ldots, \epsilon_n p_n\}$. In particular $p_{n+1} \in T'$ or $\neg p_{n+1} \in T'$. Without loss of generality assume that p_{n+1} (i.e. $0 p_{n+1}$) belongs to T. Setting $\epsilon_{n+1} = 0$ we have

$$\langle \epsilon_0 p_0, \ldots, \epsilon_{n+1} p_{n+1} \rangle \in \mathit{tree}^W.$$

Hence, the default rule $\frac{:\epsilon_0 p_0 \wedge \ldots \wedge \epsilon_{n+1} p_{n+1}}{\epsilon_0 p_0 \wedge \ldots \wedge \epsilon_{n+1} p_{n+1}}$ belongs to D. Since $T \subseteq T'$, p_{n+1} is consistent with T. Therefore $\epsilon_0 p_0 \wedge \ldots \wedge \epsilon_{n+1} p_{n+1}$ is consistent with T and thus $\epsilon_0 p_0 \wedge \ldots \wedge \epsilon_{n+1} p_{n+1}$ belongs to T. In particular $p_{n+1} \in T$, a contradiction. Therefore the set of generating default rules for T is infinite and determines a branch through tree^W. Thus T is a complete theory containing W. This completes the proof. $\square$

6 Conclusions

The concepts studied in this paper, representability and equivalence, are of key importance for default logic and its applications. Representability provides insights into the expressive power of default logic, while equivalence provides normal form results for default logic, allowing the user to find simpler representations for his/her default theories.

In this paper we characterized those families of theories that can be represented by default theories with a finite set of defaults (Theorem 3.2 and Corollary 3.3). However, we have not found a characterization of families of theories that are representable by default theories with an infinite set of defaults. This problem seems to be much more difficult and remains open. In the paper, we present two countable families that are not representable and completely solved the representability problem if inifintary defaults are allowed.

We also studied representability by means of normal default theories. Here, our results are complete. Corollary 5.2 provides a full description of families of theories that are collections of extensions of normal default theories.

Another notion studied in the paper was equivalence of default theories. We showed (Theorem 3.1) that for every normal default theory there exists a normal default theory consisting of prerequisite-free defaults and having exactly the same extensions as the original one. We also exhibited some cases when, for a given normal default theory, an equivalent normal default theory can be found with empty objective part. Finding a complete description of normal default theories for which it is possible remains an open problem.

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