A Context for Belief Revision: Forward Chaining-Normal Nonmonotonic Rule Systems:

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1 Introduction and Motivation.

A number of nonmonotonic reasoning formalism have been introduced to model the set of beliefs of an agent. For example, Reiter [Rei80] introduced default logic where the set of beliefs of an agent reasoning with incomplete information corresponded to a extension of a default theory $< D, W >$. In the realm of logic programming, the stable models of a general logic program as introduced by Gelfond and Lifschitz [GL88] can be used to model the set of beliefs of an agent. Similarly, extensions of truth maintenance systems as defined by Doyle [Doy79] and De Kleer [dK86], with subsequent contributions of Reinfrank, Dressler, and Brewka [RDB89] can be used to model the set of beliefs of an agent. We introduced nonmonotonic rule systems as a non-logical generalization of all essential features of default theories, truth maintenance systems, and logic programs so that theorems applying to all could be proven once and for all, see [MNR90] and [MNR92c].

We put forth the hypothesis that any of the above systems can be used to effectively model the beliefs of an agent as long as we restrict ourselves to a rather wide class of default theories, general logic programs, or truth maintenance systems which correspond to the forward chaining -normal nonmonotonic rule systems, or FC-normal systems, introduced in this paper. For example, suppose that belief sets are identified with stable models of an FC-normal logic program. To explain the role of FC-normality, we employ the paradigm of a blocks world for a robot with a hardwired motion planner. We think of the robot as making moves based on facts about where blocks are and based on rules of thumb such as "as long as such and such configuration is not observed, move as follows" based on logic programming. We choose as our current belief set a stable model, if any, incorporating known facts and rules. First difficulty: a general logic program may have no stable model. A second difficulty can arise even if our logic program has a stable model. That is, we want to *deduce* robot moves in this stable model. But if the robot observes a new fact, a block

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position not in the stable model, we have to abandon the previous stable model of the old program and find a new stable model of the logic program extended by the new facts. Again, for arbitrary logic programs, there may be no such stable model. So if stable models are to model beliefs and the robot is to have a belief set no matter what facts arise, with a hardwired underlying logic program, we have to find a useful and general condition on the logic program which guarantees that the program, extended by new facts, always has a stable model. Such a condition is the irreducible minimum in order to model belief sets as stable models and to specify belief revision operations as programming operations on stable models.

In default logic, Reiter [Rei80] (see also Etherington ([Eth88]) defined a simple class of default rules, called *normal*, for which one never gets stuck in finding extensions. His normality condition is syntactic and unduly restrictive, i.e. in a normal default theory all rules must be of the form $\frac{\phi:M\beta}{\beta}$. It turns out that this syntactic condition does not generalize directly to nonmonotonic rule systems. However an analysis of the proofs of the main results of Reiter on normal default theories reveals that his proofs do not rely on the particular syntactic form of his rules but rather on the fact that all rules of the form $\frac{\phi:M\beta}{\beta}$ have a certain consistency property. This led us to define a far reaching generalization of normal default theories with respect to a general consistency property. Moreover this generalization is easily extendible to nonmonotonic rule systems and hence applies to general logic programs and truth maintenance systems as well. Thus we introduce in this paper the definition of what we call FC-normal nonmonotonic rule systems. We shall see that when we translate FC-normal nonmonotonic rule systems back into default theories, we will define a large class of default theories which we call FC-normal default theories. We shall see that the class of FC-normal default theories strictly contains the class of normal default theories and that FC-normal default theories have all the desirable properties of normal default theories. Finally there are natural analogues of FC-normal default theories or FC-normal nonmonotonic rule systems in the formalisms of logic programming and truth maintenance systems as well.

To repeat, we claim that extensions of nonmonotonic rule systems can be used to model beliefs of an agent, provided that we restrict ourselves to the class of FC-normal nonmonotonic rule systems introduced in this paper. Here an extension of a nonmonotonic rule system is the common generalization of extensions of default theories, stable models of logic programs, and extensions of truth maintenance systems. Restricting ourselves to FC-normal nonmonotonic rule systems avoids our ever getting stuck in finding new extensions in the face of new facts and there is no syntactic limitation. Rather, we employ an axiomatic "consistency property", different for each application, which cover all areas of intended application we have examined. Our notion of consistency property can be thought of as a version of Scott's "information systems" [Sco82] tailored to extensions of nonmonotonic rule systems. In future papers this will allow us to use, for example, metaprogramming as a belief revision programming language operating on "FC-normal" logic programs and representations of stable models. Now the problem of whether there exists a stable model of a finite propositional program is known to be NP-complete ([MT91]). The same result applies to the problem of whether there exists an extension of a nonmonotonic rule system. So the problem of finding a consistency property under which a logic program P or a nonmonotonic rule system $\langle U, N\rangle$ is FC-normal is at least, NP-hard. But in all applications, we can see directly what consistency notion to introduce, dictated by natural considerations of consistency for the semantics of the application.

We shall not only prove that FC-normal nonmonotonic rule system have the desirable properties possessed by normal default theories but we shall prove that FC-normal nonmonotonic rule systems have a number of other important properties as well. For example, given an FC-normal nonmonotonic rule system $\mathcal{S} = \langle U, N\rangle$, we shall show that every extension of $\mathcal{S}$ can be constructed via a simple forward chaining algorithm which is based on a well-ordering $\prec$ of the strictly nonmonotonic rules of $\mathcal{S}$ which we denote by $nmon(\mathcal{S})$. That is, each ordering of $nmon(\mathcal{S})$ determines an extension of $\mathcal{S}$ and every extension of $\mathcal{S}$ is determined by an ordering $\prec$. However, many orderings may determine the same extension. This means the orderings themselves may be taken as computational surrogates for extensions. In a companion paper [MNR93b], we introduce a more general forward chaining process based on a well-ordering of the nonmonotonic rules which can be applied to arbitrary nonmonotonic rule systems $\mathcal{S}$. For general nonmonotonic rule systems, our forward chaining process also yields an extension but, for a possibly smaller rule system than our original rule system. It turns out that one can view our definition of FC-normal nonmonotonic rule systems as a sufficient condition which guarantees that the forward chaining algorithm always produces an extension of the original rule system. This is important because we will show that the forward chaining algorithm, when applied to finite FC-normal nonmonotonic rule systems, produces an extension in polynomial time in the sum of the lengths of the rules of the system. More precisely one can construct an extension of an FC-normal nonmonotonic rule system in time which is of order the square of the sum of the lengths of the rules of system. As usual, this same result also applies to constructing extensions of FC-normal default theories, stable models of "FC-normal" logic programs, and extensions of "FC-normal" truth maintenance systems. Thus FC-normal nonmonotonic rules system have the property that one can construct an extension or go from one extension to another in a highly efficient manner.

Finally we explore the complexity of the sets of extensions of arbitrary recursive FC-normal nonmonotonic rule systems. Briefly, a nonmonotonic rule system $\mathcal{S} = \langle U, N\rangle$ is called recursive if the universe U is a recursive set of integers and the set of rules N is a recursive set. We showed in [MNR92b] that for every countably branching recursive tree, there is a recursive nonmonotonic rule system $\mathcal{S} = \langle U, N\rangle$ such that there is a recursive one-to-one correspondence between maximal branches of the tree and the extensions of the $\mathcal{S}$. Conversely, the set of all extensions of a recursive nonmonotonic rule system always so arises. Since there are recursive trees without hyperarithmetic maximal branches, but with a continuum of maximal branches, it follows there are recursive NRS with a continuum of extensions but no hyperarithmetic extensions. This does not happen with FC-normal recursive NRS. They always have extensions recursive in $0''$. However, recursive FC-normal nonmonotonic rule systems are very expressive. We show that given any highly recursive tree, i.e. a recursive tree which is finitely branching and which has the property that we can effectively find all the successors of any node in the tree, there is an FC-normal nonmonotonic rule system $\mathcal{S} = \langle U, N \rangle$ such that there is a recursive one-to-one correspondence between maximal branches of the tree and the extensions of the $\mathcal{S}$. This is shown via coding trees into suitably constructed recursive NRS with simultaneous construction of a suitable consistency property. This will imply that, for example, that there are recursive FC-normal nonmonotonic rules systems which have no recursive extensions.

The outline of this paper is as follows. In Section 2, we shall briefly review Reiter's [Rei80] definition of normal default theories and state the main properties of such theories. Then in Section 3 we introduce the basic definitions of nonmonotonic rule systems. In Section 4, we shall introduce our abstract consistency properties and define FC-normal nonmonotonic rules systems. We shall also introduce our forward chaining construction and state our major results about FC-normal nonmonotonic rules systems. The proofs of these theorems will be postponed until Section 7. In Section 5, we shall show how to translate the definitions and results of Section 4 back into logic programming, default logic, and truth maintenance systems. In Section 6, we shall provide a background on the complexity of general recursive nonmonotonic rule systems and state our basic complexity results for recursive FC-normal nonmonotonic rule systems. The proofs of the complexity results in Section 6 will be postponed until Section 8.

2 Normal Default Theories.

In this section we shall introduce Reiter's definitions of default theories and normal default theories and state some of the basic theorems about normal default theories as proved in [Rei80].

Following the notation of Reiter's paper ([Rei80]), a *default rule* is a rule of proof of the form

$$\frac{\varphi : M\psi_1, \ldots, M\psi_m}{\gamma} \tag{1}$$

where $\varphi, \psi_1 \ldots, \psi_m, \gamma$ are formulas of a propositional language $\mathcal{L}$. A *default theory* is a pair $\langle D, W \rangle$ where D is a set of default rules and $W \subseteq \mathcal{L}$. For any subset of formulas $S \subseteq \mathcal{L}$, we let $Cn(S)$ denote the set of all logical consequences of S. Also if

D is a set of default rules, let

$$c(D) = \{\gamma : \frac{\varphi : M\psi_1, \ldots, M\psi_m}{\gamma} \in D\}.$$

Given a subset $S \subseteq \mathcal{L}$, define $\Gamma(S)$ as the least set T (under inclusion) satisfying these conditions:

1. $W \subseteq T$;
2. $Cn(T) = T$;
3. Whenever $r \in D$ is a default rule of the form (1) and $\varphi \in T$ and for all $j \leq m$, $\neg\psi_j \notin S$ then $\gamma \in T$.

It is easy to see that $\Gamma(S)$ always exists. We say that $S \subseteq \mathcal{L}$ is an *extension* of $\langle D, W\rangle$ if $\Gamma(S) = S$. A default rule of the form (1) is called *generating* for S if $\varphi \in S$, $\neg\psi_1, \ldots, \neg\psi_m \notin S$. Let $S \subseteq \mathcal{L}$. Then we define $GD(D, S)$ as the set of all generating rules for S in D and $c(GD(D, S))$ is the set of their conclusions.

A rule r is **normal** if it is of the form

$$\frac{\varphi : M\psi}{\psi}. \tag{2}$$

A default theory $\langle D, W\rangle$ is **normal** if every $r \in D$ is a normal default rule. Reiter [Rei80] proved the following theorems about normal default theories.

Theorem 2.1 *Every normal default theory possesses an extension.*

Theorem 2.2 (Semi-monotonicity) *Suppose that D and D' are sets of normal defaults with $D' \subseteq D$. Let E' be an extension of the normal default theory $\Delta' = \langle D', W\rangle$ and let $\Delta = \langle D, W\rangle$. Then Δ has an extension E such that*

1. *$E' \subseteq E$ and*
2. *$GD(E', \Delta') \subseteq GD(E, \Delta)$.*

Theorem 2.3 (Orthogonality of Extensions) *If a normal default theory $\langle D, W\rangle$ has distinct extensions E and F, then $E \cup F$ is inconsistent.*

Theorem 2.4 *Suppose that $\Delta = \langle D, W\rangle$ is a normal default theory and $W \cup c(D)$ is consistent. Then Δ has a unique extension.*

Theorem 2.5 *Suppose that $\Delta = \langle D, W\rangle$ is a normal default theory and that $D' \subseteq D$. Suppose further that ${E_1}'$ and ${E_2}'$ are distinct extensions of $\langle D', W\rangle$. Then Δ has distinct extensions E_1 and E_2 such that ${E_1}' \subseteq E_1$ and ${E_2}' \subseteq E_2$.*

3 Monotonic and Nonmonotonic rule systems.

In this section we recall the basic definitions of monotonic and nonmonotonic rule systems from [MNR90, MNR92c].

Tarski [Tar56] characterized monotonic formal systems by means of monotonic rules of inference. Such systems include intuitionistic logic, classical logics, modal logics, and many others. Suppose that a nonempty set U is given. In a particular application U may be the collection of all formulas of a propositional or first order logic, of all legal strings of a formal system, or of all atomic statements as in logic programming.

A **monotonic rule of inference** is a tuple $r = \langle P, \varphi \rangle$ where $P = \langle \alpha_1, \ldots, \alpha_n \rangle$ is a finite (possibly empty) list of objects from U and φ is an element of U. Such a rule r is usually written in the suggestive form

$$r = \frac{\alpha_1, \ldots, \alpha_n}{\varphi} \tag{3}$$

We call $\alpha_1, \ldots, \alpha_n$ the premises of r and φ the conclusion of r.

Definition 3.1 *(a) A monotonic formal system is a pair $\langle U, M \rangle$, where U is a nonempty set and M is a collection of monotonic rules.*
(b) A subset $S \subseteq U$ is called **deductively closed** *over $\langle U, M \rangle$ if for all rules $r = \frac{\alpha_1, \ldots, \alpha_n}{\varphi} \in M$, $\alpha_1, \ldots, \alpha_n \in S$ implies $\varphi \in S$.*

Inspired by Reiter [Rei80], and Apt [Apt90], we introduced the notion of a nonmonotonic formal system $\langle U, N \rangle$ in [MNR90, MNR92c]. A **nonmonotonic rule of inference** is a triple $\langle P, R, \varphi \rangle$, where $P = \{\alpha_1, \ldots, \alpha_n\}$, $R = \{\beta_1, \ldots, \beta_m\}$ are finite lists of objects from U and $\varphi \in U$. Each such rule is written in a more suggestive form as

$$r = \frac{\alpha_1, \ldots, \alpha_n : \beta_1, \ldots, \beta_m}{\varphi}. \tag{4}$$

Here $\alpha_1, \ldots, \alpha_n$ are called the *premises* of rule r, $\beta_1, \ldots, \beta_m$ are called the *constraints* of rule r, and φ is called the *conclusion* of rule r. For any rule r as in (4), we shall write $prem(r) = \{\alpha_1, \ldots, \alpha_n\}$, $cons(r) = \{\beta_1, \ldots, \beta_m\}$, and $c(r) = \varphi$. Either $prem(r)$, or $cons(r)$, or both may be empty. If $prem(r) = cons(r) = \emptyset$, then the rule r is called an **axiom**.

A **nonmonotonic rule system** is a pair $\langle U, N \rangle$, where U is a non-empty set and N is a set of nonmonotonic rules such that $prem(r), cons(r)$, and $\{c(r)\}$ are subsets of U for all $r \in N$.

Each monotonic formal system can be identified with the nonmonotonic system in which every monotonic rule has an empty set of constraints.

A subset $S \subseteq U$ is called **deductively closed** if for all $r = \frac{\alpha_1,\ldots,\alpha_n:\beta_1,\ldots,\beta_m}{\varphi} \in N$, whenever all the premises $\alpha_1, \ldots, \alpha_n$ of r are in S and all the constraints $\beta_1, \ldots, \beta_m$ of r are not in S, then the conclusion φ of r belongs to S.

In nonmonotonic systems, deductively closed sets are not generally closed under arbitrary intersections as in the monotone case. But deductively closed sets are closed under intersections of descending chains. Since U is deductively closed, by the Kuratowski-Zorn Lemma, any $I \subseteq U$, there is at least one minimal deductively closed set superset of I.

Given sets $S \subseteq U$ and $I \subseteq U$, an S-**deduction** of φ from I in $\langle U, N\rangle$ is a finite sequence $< \varphi_1, \ldots, \varphi_k >$ such that $\varphi_k = \varphi$ and, for all $i \leq k$, each φ_i is in I, or is an axiom, or is the conclusion of a rule $r \in N$ such that all the premises of r are included in $\{\varphi_1, \ldots, \varphi_{i-1}\}$ and all constraints of r are in $U - S$ (see [MT89b], also [RDB89]). An S-**consequence** of I is an element of U occurring in some S-deduction from I. Let $C_S(I)$ be the set of all S-consequences of I in $\langle U, N\rangle$. Clearly I is a subset of $C_S(I)$. However note that S enters solely as a restraint on the use of the rules imposed by the constraints in the rules. A single constraint in a rule in N may be in S and thus prevent the rule from ever being applied in an S-deduction from I, even though all the premises of that rule occur earlier in a deduction. Thus S contributes no members directly to $C_S(I)$, although members of S may turn up in $C_S(I)$ by an application of a rule which happens to have its conclusion in S. For a fixed S, the operator $C_S(\cdot)$ is monotonic. That is, if $I \subseteq J$, then $C_S(I) \subseteq C_S(J)$. Also, $C_S(C_S(I)) = C_S(I)$. However, for fixed I, the operator $C_S(I)$ is anti-monotonic in the argument S. That is if $S' \subseteq S$, then $C_S(I) \subseteq C_{S'}(I)$.

Generally, $C_S(I)$ is not deductively closed in $\langle U, N\rangle$. It is perfectly possible that all the premises of a rule be in $C_S(I)$, the constraints of that rule are outside $C_S(I)$, but a constraint of that rule be in S, preventing the conclusion from being put into $C_S(I)$.

Example 3.1 $U = \{\alpha, \beta, \gamma\}, N = \{\frac{:}{\alpha}, \frac{\alpha:\beta}{\gamma}\}$, $S = \{\beta\}$. Then $C_S(\emptyset) = \{\alpha\}$ is not deductively closed.

However, the following holds:

Proposition 3.2 *If $S \subseteq C_S(I)$ then $C_S(I)$ is deductively closed.*

We say that $S \subseteq U$ is **grounded** in I if $S \subseteq C_S(I)$. We say that $S \subseteq U$ is an **extension** of I if $C_S(I) = S$.

The notion of groundedness is related to the phenomenon of "reconstruction". S is grounded in I if all elements of S are S-deducible from I (remember that S influences only the negative sides of rules). S is an extension of I if two things happen.

First, every element of S is deducible from I, that is, S is grounded in I (this is an analogue of adequacy property in logical calculi). Second, the converse holds: all the S-consequences of I belong to S (this is the analogue of completeness). Thus extensions are analogues for a nonmonotonic systems of the set of all consequences for monotonic systems. Both properties (adequacy and completeness) need to be satisfied - if we want S to be an extension.

The notion of an extension is related to that of a minimal deductively closed set. Indeed, the following propositions are proved in [MNR90]

Proposition 3.3 *If S is an extension of I, then:*
(1) S is a minimal deductively closed superset of I.
(2) For every I' such that $I \subseteq I' \subseteq S$, $C_S(I') = S$.

Proposition 3.4 *The set of extensions of I forms an antichain. That is, if S_1, S_2 are extensions of I and $S_1 \subseteq S_2$, then $S_1 = S_2$.*

With each rule r of form (4), we associate a monotonic rule of form (3)

$$r' = \frac{\alpha_1, \ldots, \alpha_n}{\varphi} \tag{5}$$

obtained from r by dropping all the constraints. Rule r' is called the *projection* of rule r. Let $NG(S, \mathcal{S})$ be the collection of all S-applicable rules. That is, a rule r belongs to $NG(S, \mathcal{S})$ if all the premises of r belong to S and all constraints of r are outside of S. We write $M(S)$ for the collection of all projections of all rules from $NG(S, \mathcal{S})$. The projection $\langle U, N \rangle \mid_S$ is the monotone system $\langle U, M(S) \rangle$. Thus $\langle U, N \rangle \mid_S$ is obtained as follows: First, non-S-applicable rules are eliminated. Then, the constraints are dropped altogether. We have the following characterization theorem:

Theorem 3.5 *A subset $S \subseteq U$ is an extension of I in $\langle U, N \rangle$ if and only if S is the deductive closure of I in $\langle U, N \rangle \mid_S$.*

For the rest of this paper, we shall only consider extensions of $\emptyset$ unless explicitly stated otherwise. We say that T is an *extension* of $\mathcal{S}$ if T is an extension of $\emptyset$ in $\mathcal{S}$.

We shall end this section by giving yet another characterization of extensions. For this we need the concept of a proof scheme. A *proof scheme* for φ is a finite sequence

$$p = << \varphi_0, r_0, G_0 >, \ldots, < \varphi_m, r_m, G_m >> \tag{6}$$

such that $\varphi_m = \varphi$ and
(1) If $m = 0$ then:

(a) φ_0 is an axiom (that is, there exists a rule $r \in N$ such that $r = \frac{:}{\varphi_0}$), $r_0 = r$, and $G_0 = \emptyset$,
or
(b) φ is a conclusion of a rule $r = \frac{:b_1,\ldots,b_r}{\varphi}$, $r_0 = r$, and $G_0 = cons(r)$.
(2) If $m > 0$, $<< \varphi_i, r_i, G_i >>_{i=0}^{m-1}$ is a proof scheme of length m and φ_m is a conclusion of $r = \frac{\varphi_{i_0},\ldots\varphi_{i_s}:b_1,\ldots,b_r}{\varphi_m}$ where $i_0, \ldots, i_s < m$, $r_m = r$, and $G_m = G_{m-1} \cup cons(r)$.
The formula φ_m is called the *conclusion* of p and is written $cln(p)$. The set G_m is called the *support* of p and is written $supp(p)$.

The idea behind this concept is as follows. An S-derivation in the system $\langle U, N\rangle$, say p, uses some negative information about S to ensure that the constraints of rules that were used are outside of S. But this negative information is finite, that is, it involves a finite subset of the complement of S. Thus, there exists a finite subset G of the complement of S such that as long as $G \cap S_1 = \emptyset$, p is an S_1-derivation as well. Our notion of proof scheme captures this finitary character of S-derivation.

We can then characterize extensions of $\langle U, N\rangle$ as follows.

Theorem 3.6 *Let $\mathcal{S} = \langle U, N\rangle$ be a nonmonotonic rule system and let $S \subset U$. Then S is an extension of $\mathcal{S}$ if and only if*
(i) for each $\varphi \in S$, there is a proof scheme p such that $cln(p) = \varphi$ and $supp(p) \cap S = \emptyset$ and
(ii) for each $\varphi \notin S$, there is a no proof scheme p such that $cln(p) = \varphi$ and $supp(p) \cap S = \emptyset$.

There is a natural preordering of proof schemes according to the set of rules they utilize. Given a proof scheme p, there is always a minimal (with respect to that preordering) proof scheme with the same conclusion. This fact will be used in our recursion-theoretic considerations. The concept of minimal proof scheme is treated in more detail in [MNR90, MNR92c]

4 FC-normal Nonmonotonic Rule Systems.

In this section we shall define forward chaining-normal nonmonotonic rule systems (FC-normal nonmonotonic rule systems) and state the analogues of all the theorems of the previous section. We shall postpone the proof of all theorems in this section until section 7

Let $\mathcal{S} = \langle U, N\rangle$ be a nonmonotonic rule system. Let $mon(\mathcal{S})$ be the set of all rules $r \in N$ such that r has no constraints. Thus $mon(\mathcal{S}) = \{r \in N : cons(r) = \emptyset\}$. We let $nmon(\mathcal{S}) = N - mon(\mathcal{S})$. We shall refer to $mon(\mathcal{S})$ as the **monotonic part** of $\mathcal{S}$ and $nmon(\mathcal{S})$ as the **nonmonotonic part** of $\mathcal{S}$. We say a set $W \subseteq U$ is **monotonically**

closed if whenever $r = \frac{\alpha_1, \ldots, \alpha_n}{\gamma} \in mon(\mathcal{S})$ and $\alpha_1, \ldots, \alpha_n \in W$, then $\gamma \in W$. Given any set $A \subseteq U$, the **monotonic–closure** of A, written $cl_{mon}(A)$, is defined to be the intersection of all monotonically closed sets containing A. It is easy to see that such a set is itself monotonically closed.

So far our investigations of nonmonotonic rule systems did not get beyond the information established already in [MNR90, MNR92c]. Next we introduce the notion of consistency property over $\langle U, N\rangle$, which leads us to the main subject of this paper.

We say that a subset $Con \subseteq \mathcal{P}(U)$ (where $\mathcal{P}(U)$ is the power set of U) is a **consistency property** over $\mathcal{S} = \langle U, N\rangle$ if

1. $\emptyset \in Con$;
2. $\forall A, B \subseteq U(A \subseteq B \quad \wedge \quad Con(B) \Rightarrow Con(A))$;
3. $\forall A \subseteq U(Con(A) \Rightarrow Con(cl_{mon}(A)))$;
4. whenever $\mathcal{A} \subseteq Con$ has the property that $A, B \in \mathcal{A} \Rightarrow \exists_{C \in \mathcal{A}}(A \subseteq C \wedge B \subseteq C)$, then $Con(\bigcup \mathcal{A})$.

Condition (1) says that the empty set is consistent. Condition (2) requires that a subset of a consistent set is also consistent. Condition (3) postulates that the closure of a consistent set under monotonic rules is consistent. Finally, the last condition says that the union of a *directed* family of consistent sets is also consistent. We note that conditions (1),(2), and (4) are Scott's conditions for information systems. Condition (3) connects "consistent" sets to the monotonic part of the rule system; if A is consistent then adding elements derivable from A via monotonic rules preserves "consistency".

Now suppose $\mathcal{S} = \langle U, N\rangle$ is a nonmonotonic rule system and let Con be a consistency property over $\langle U, N\rangle$. Then we say a rule $r = \frac{\alpha_1, \ldots, \alpha_n : \beta_1, \ldots, \beta_t}{\gamma} \in nmon(\mathcal{S})$ is **FC-normal** (with respect to Con) if $Con(V \cup \{\gamma\})$ and not $Con(V \cup \{\gamma, \beta_i\})$ for all $i \leq k$ whenever $V \subseteq U$ is such that $Con(V)$, $cl_{mon}(V) = V$, $\alpha_1, \ldots, \alpha_n \in V$, and $\gamma, \beta_1, \ldots, \beta_k \notin V$. We say that $\mathcal{S} = \langle U, N\rangle$ is a **FC-normal** (with respect to Con) if all $r \in nmon(\mathcal{S})$ are FC-normal with respect to Con. Finally, we say that $\langle U, N\rangle$ is **FC-normal nonmonotonic rule system** if for some consistency property $Con \subseteq \mathcal{P}(U)$, $\langle U, N\rangle$ is FC-normal with respect to Con.

Example 4.1 Let $U = \{a, b, c, d, e, f\}$. Let the consistency property be defined by the following condition:
$A \notin Con$ if and only if either $\{c, d\} \subseteq A$ or $\{e, f\} \subseteq A$.

Thus $\{a,b,c,e\}$, $\{a,b,c,f\}$, $\{a,b,d,e\}$, and $\{a,b,d,f\}$ are the maximal subsets of $\mathcal{P}(U)$ which are in Con.

Now consider the following set of rules, N:

(1) $\frac{:}{a}$
(2) $\frac{c:}{b}$
(3) $\frac{b:}{c}$
(4) $\frac{a:d}{c}$
(5) $\frac{c:f}{e}$

Then for the nonmonotonic rule system $\mathcal{S} = \langle U, N\rangle$, rules (1),(2), and (3) form the monotonic part of $\mathcal{S}$ and rules (4) and (5) form the nonmonotonic part of $\mathcal{S}$. First it is easy to check that Con is a consistency property over $\mathcal{S}$. The monotonically closed subsets of $\mathcal{P}(U)$ which are in Con are $\{a\}$,$\{a,d\}$,$\{a,e\}$,$\{a,f\}$,$\{a,b,c\}$,$\{a,d,e\}$, $\{a,d,f\}$, $\{a,b,c,e\}$,and $\{a,b,c,f\}$. It is then easy to check that $\mathcal{S}$ is FC-normal with respect to Con. Moreover one can easily check that $\mathcal{S}$ has a unique extension $M = \{a,b,c,e\}$.

If we add to N the rule $\frac{c:}{d}$ to get a set of rules N', then Con is no longer a consistency property over $\mathcal{S}' = \langle U, N'\rangle$ because $\{c\} \in Con$ but the monotonic closure of $\{c\}$ relative to $\mathcal{S}' = \langle U, N'\rangle$ which equals $\{a,b,c,d\}$ is not in Con.

If we add the rule $\frac{e:f}{d}$ to N to form a new NRS $\mathcal{S}'' = \langle U, N''\rangle$, Con will still be a consistency property over $\mathcal{S}'' = \langle U, N''\rangle$ because the property of being a consistency property depends only on the monotonic part of the rule system. However $\mathcal{S}'' = \langle U, N''\rangle$ is not FC-normal with respect to Con because $r = \frac{e:f}{d}$ is not FC-normal with respect to Con. That is, for the monotonically closed set $\{a,b,c,e\}$, we have $prem(r) \subseteq \{a,b,c,e\}$, $cons(r) \cap \{a,b,c,e\} = \emptyset$, but $cl_{mon}(\{c(r)\} \cup \{a,b,c,e\}) = \{a,b,c,d,e\}$ is not in Con.

Finally if we add to N the rule $\frac{c:e}{f}$ to get a set of rules N''', then the resulting NRS $\mathcal{S}''' = \langle U, N'''\rangle$ is still FC-normal with respect to Con but now there are two extensions, $M_1 = \{a,b,c,e\}$ and $M_2 = \{a,b,c,f\}$.

We have the following analogue of Theorem 2.1.

Theorem 4.1 *Let $\mathcal{S} = \langle U, N\rangle$ be an FC-normal nonmonotonic rule system with respect to consistency property Con. Then there exists an extension of $\mathcal{S}$.*

It is easy to adopt the proof of Theorem 4.1 to get the following result.

Theorem 4.2 *Let $\mathcal{S} = \langle U, N\rangle$ be a normal nonmonotonic rule system with respect to consistency property Con and let I be a subset of U such that $I \in Con$. Then there exists an extension I' of $\mathcal{S}$ such that $I \subseteq I'$.*

In fact, we will show that there is a uniform construction of extensions of FC-normal NRS $\mathcal{S} = \langle U, N \rangle$ which depends on a well-ordering $\prec$ of the nonmonotonic rules of $\mathcal{S}$, i.e. the rules in $nmon(\mathcal{S})$. We shall call this construction the **forward chaining** construction with respect to $\prec$ (and this is the reason why we call our systems, for whom this construction always succeeds in producing an extension, forward chaining normal). To this end, fix some well-ordering $\prec$ of $nmon(\mathcal{S})$. That is, the well-ordering $\prec$ determines some listing of the rules of $nmon(\mathcal{S})$,$\{r_\alpha : \alpha \in \gamma\}$ where γ is some ordinal. Let Θ_γ be the least cardinal such that $\gamma \leq \Theta_\gamma$. In what follows, we shall assume that the ordering among ordinals is given by $\in$. Our forward chaining construction will define an increasing sequence of sets $\{E^{\prec}_\alpha\}_{\alpha \in \Theta_\gamma}$. We will then define $E^{\prec} = \bigcup_{\alpha \in \Theta_\gamma} E^{\prec}_\alpha$ and show that $E^{\prec}$ is always an extension of $\mathcal{S}$. Moreover we shall show that all extensions of $\mathcal{S}$ arise from this construction.

The forward chaining construction of $E^{\prec}$.

<u>Case 0</u>. Let $E^{\prec}_0 = cl_{mon}(\emptyset)$.

<u>Case 1</u>. $\alpha = \eta + 1$ is a successor ordinal.
Given $E^{\prec}_\eta$, let $\ell(\alpha)$ be the least $\lambda \in \gamma$ such that $r_\lambda = \dfrac{\alpha_1, \ldots, \alpha_p : \beta_1, \ldots, \beta_k}{\psi}$ where $\alpha_1, \ldots, \alpha_p \in E^{\prec}_\eta$ and $\beta_1, \ldots, \beta_k$, $\psi \notin E^{\prec}_\eta$. If there is no such $\ell(\alpha)$, then let $E^{\prec}_{\eta+1} = E^{\prec}_\alpha = E^{\prec}_\eta$. Otherwise, let

$$E^{\prec}_{\eta+1} = E^{\prec}_\alpha = cl_{mon}(E^{\prec}_\eta \cup \{cln(r_{\ell(\alpha)})\}).$$

<u>Case 2</u>. α is a limit ordinal. Then let $E^{\prec}_\alpha = \bigcup_{\beta \in \alpha} E^{\prec}_\beta$.

This given, we have the following.

Theorem 4.3 *If $\mathcal{S} = \langle U, N \rangle$ is an FC-normal nonmonotonic rule system and $\prec$ is* any *well-ordering of $nmon(\mathcal{S})$, then*

1. *$E^{\prec}$ is an extension of $\mathcal{S}$.*

2. *(Completeness of the construction). Every extension of $\mathcal{S}$ is of the form $E^{\prec}$ for a suitably chosen ordering $\prec$ of $nmon(\mathcal{S})$.*

It is quite straightforward to prove by induction that if $\mathcal{S} = \langle U, N \rangle$ is FC-normal with respect to consistency property Con, then $E^{\prec}_\alpha \in Con$ for all α and hence $E^{\prec} \in Con$. Thus the following is an immediate consequence of Theorem 4.3(2).

Corollary 4.4 *Let $\mathcal{S} = \langle U, N\rangle$ be an FC-normal nonmonotonic rule system with respect to consistency property Con, then every extension of $\mathcal{S}$ is in Con.*

Example 4.2 If we consider the final extended program of Example 4.1, it is easy to check that any ordering $\prec_1$ in which the rule $r_1 = \frac{c:f}{e}$ precedes the rule $r_2 = \frac{c:e}{f}$ will have $E^{\prec_1} = M_1$ while any ordering $\prec_2$ in which r_2 precedes r_1 will have $E^{\prec_2} = M_2$.

We should also point out that if we restrict ourselves to countable nonmonotonic rules systems $\mathcal{S} = \langle U, N\rangle$, i.e. if U and N are countable, then we can restrict ourselves to orderings of order type ω where ω is the order type of the natural numbers. That is, suppose we fix some well-ordering $\prec$ of $nmon(\mathcal{S})$ of order type ω. Thus, the well-ordering $\prec$ determines some listing of the rules of $nmon(\mathcal{S})$,$\{r_n : n \in \omega\}$. Our forward chaining construction can be presented in a more straightforward manner in this case. Our construction again will define an increasing sequence of sets $\{E^{\prec}_n\}_{n\in\omega}$ in stages. This given, we will then define $E^{\prec} = \bigcup_{n\in\omega} E^{\prec}_n$.

The countable forward chaining construction of $E^{\prec}$.

<u>Stage 0</u>. Let $E^{\prec}_0 = cl_{mon}(\emptyset)$.

<u>Stage $n+1$</u>. Let $\ell(n+1)$ be the least $s \in \omega$ such that $r_s = \dfrac{\alpha_1, \ldots, \alpha_p : \beta_1, \ldots, \beta_k}{\psi}$ where $\alpha_1, \ldots, \alpha_p \in E^{\prec}_n$ and $\beta_1, \ldots, \beta_k$, $\psi \notin E^{\prec}_n$. If there is no such $\ell(n+1)$, then let $E^{\prec}_{n+1} = E^{\prec}_n$. Otherwise, let

$$E^{\prec}_{n+1} = E^{\prec}_n = cl_{mon}(E^{\prec}_n \cup \{cln(r_{\ell(n+1)})\}).$$

This given, we then have the following.

Theorem 4.5 *If $\mathcal{S} = \langle U, N\rangle$ is a countable FC-normal nonmonotonic rule system, then*

1. *$E^{\prec}$ is an extension of $\mathcal{S}$ if $E^{\prec}$ is constructed via the countable forward chaining algorithm with respect to $\prec$ where $\prec$ is* any *well-ordering of $nmon(\mathcal{S})$ of order type ω.*

2. *(completeness of the construction.) Every extension of $\mathcal{S}$ is of the form $E^{\prec}$ for a suitably chosen well ordering $\prec$ of $nmon(\mathcal{S})$ of order type ω where $E^{\prec}$ is constructed via the countable forward chaining algorithm.*

FC-normal NRS's also possess the "semi-monotonicity" property.

Theorem 4.6 *Let $\mathcal{S}_1 = \langle U, N_1 \rangle$ and $\mathcal{S}_2 = \langle U, N_2 \rangle$ be two FC-normal NRS such that $N_1 \subseteq N_2$ but $mon(\mathcal{S}_1) = mon(\mathcal{S}_2)$. Assume, in addition, that both are FC-normal with respect to the same consistency property. Then for every extension E_1 of $\mathcal{S}_1$, there is an extension E_2 of $\mathcal{S}_2$ such that*

1. *$E_1 \subseteq E_2$ and*
2. *$NG(E_1, \mathcal{S}_1) \subseteq NG(E_2, \mathcal{S}_2)$.*

Here given and extension E, we let $NG(E, \mathcal{S})$ denote the set of all E-applicable rules, i.e. $r = \dfrac{\alpha_1, \ldots, \alpha_p : \beta_1, \ldots, \beta_k}{\psi}$ is in $NG(E, \mathcal{S})$ if and only if $\alpha_1, \ldots, \alpha_p \in E$ and $\beta_1, \ldots, \beta_k$ are not in E.

FC-normal NRS's also satisfy the orthogonality of extensions property with respect to their consistency property. That is, we have the following analogue of Theorem 2.3

Theorem 4.7 *Let $\mathcal{S} = \langle U, N \rangle$ be an FC-normal NRS with respect to a consistency property Con. Then if E_1 and E_2 are two distinct extensions of $\mathcal{S}$, $E_1 \cup E_2 \notin Con$.*

Similarly we also have the following analogues of Theorems 2.4 and 2.5.

Theorem 4.8 *Let $\mathcal{S} = \langle U, N \rangle$ be an FC-normal NRS with respect to a consistency property Con. Suppose that $cl_{mon}\{cln(r) : r \in nmon(\mathcal{S})\}$ is in Con. Then $\mathcal{S}$ has a unique extension.*

We now have two more results which are also analogues of the results of Reiter's [Rei80]. We say that $\varphi \in U$ has a consistent proof scheme with respect to a consistency property Con over $\mathcal{S} = \langle U, N \rangle$ if and only if there is a proof scheme

$$p =<< \varphi_0, r_0, G_0 >, \ldots, < \varphi_m, r_m, G_m >> \tag{7}$$

such that $\varphi_m = \varphi$ and $\{\varphi_0, \ldots, \varphi_m\} \in Con$. We then have the following.

Theorem 4.9 *Let $\mathcal{S} = \langle U, N \rangle$ be an FC-normal NRS with respect to a consistency property Con. Then $\varphi \in U$ is an element of some extension of $\mathcal{S}$ if and only if φ has a consistent proof scheme with respect to Con.*

Theorem 4.10 *Suppose $\mathcal{S} = \langle U, N \rangle$ is an FC-normal NRS and that $D \subseteq nmon(\mathcal{S})$. Suppose further that E_1' and E_2' are distinct extensions of $(U, D \cup mon(\mathcal{S}))$. Then $\mathcal{S}$ has distinct extensions E_1 and E_2 such that $E_1' \subseteq E_1$ and $E_2' \subseteq E_2$.*

5 FC-normal Nonmonotonic Rule Systems and Other Nonmonotonic Reasoning Formalisms.

In this section, we shall explicitly translate the definitions and theorems of the previous section into the language of logic programming, default logic, and truth maintenance systems.

5.1 Logic programming, general case

Now because general logic programs are probably the most widely studied type of nonmonotonic reasoning we shall take some time to give the translation of FC-normal nonmonotonic rule systems and the results above into the language of logic programming. A similar translation can be done for all the other nonmonotonic formalisms to follow but we shall not carry out the translation in detail in the other cases.

A *general program clause* is an expression of the form

$$C = p \leftarrow q_1, \ldots, q_n, \neg r_1, \ldots, \neg r_m \tag{8}$$

where $p, q_1, \ldots, q_n, r_1, \ldots, r_m$ are atomic formulas possibly with variables in some first order language $\mathcal{L}$. A *program* is a set of clauses of the form (8). A clause C is called a *Horn clause* if $m = 0$. We let $H(P)$ denote the set of all Horn clauses of P.

$\mathcal{H}_P$ is the Herbrand base of P, that is, the set of all ground atomic formulas of the language of P.

$ground(P)$ is the set of ground Herbrand substitutions of clauses in P. Given a set $M \subseteq \mathcal{H}_P$, the Gelfond-Lifschitz ([GL88]) reduct of P, P^M is the set of ground Horn clauses $p \leftarrow q_1, \ldots, q_n$ such that for some $r_1, \ldots, r_m \notin M$, the clause $p \leftarrow q_1, \ldots, q_n, \neg r_1, \ldots, \neg r_m \in ground(P)$. M is called a **stable model** of P if M coincides with the least model of P^M.

Assign to a ground clause $p \leftarrow q_1, \ldots, q_n, \neg r_1, \ldots, \neg r_m \in ground(P)$ the rule

$$r(C) = \frac{q_1, \ldots, q_n : r_1, \ldots, r_m}{p}. \tag{9}$$

Let $r(P) = \langle \mathcal{H}_P, \{r(C) : C \in ground(P)\}\rangle$. Then, as shown in ([MNR90]), M is a stable model of P if and only if M is an extension of $r(P)$.

This given, it is easy to see that $\mathcal{H}_P$ plays the role of the universe U and $ground(P)$ plays the role of the set of rules N in Section 4. Moreover, the Horn part of $ground(P)$, i.e. $ground(H(P))$, plays the role of the monotonic part of the rules N. Then of course the immediate provability operator associated with $H(P)$, $T_{H(P)}$ (cf. Apt [Apt90]) is monotonic.

In this setting the notion of a consistency property becomes the following. Call a family of subsets of $\mathcal{H}_P$, Con, a **consistency property** over P if it satisfies the following conditions:

1. $\emptyset \in Con$.

2. If $A \subseteq B$ and $B \in Con$, then $A \in Con$.

3. Con is closed under directed unions.

4. If $A \in Con$ then $A \cup T_{H(P)}(A) \in Con$.

Conditions (1)-(3) are Scott's conditions for information systems. Condition (4) connects "consistent" sets of atoms to the Horn part of the program; if A is consistent then adding atoms provable from A preserves "consistency". The following fact is easy to prove:

Proposition 5.1 *If Con is a consistency property with respect to P and $A \in Con$, then $T_{H(P)} \Uparrow \omega(A) \in Con$.*

Here, for a program Q, $T_Q \Uparrow \omega(A)$ is the cumulative fixpoint of T_Q over A. Proposition 5.1 says that our condition (4) in the definition of consistency property implies that the cumulative closure of a "consistent" set of atoms under $T_{H(P)}$ is still "consistent". Here $T_Q \Uparrow \omega(A)$ is the analogue of the monotonic closure of the set A.

Given a consistency property, we define the concept of an **FC-normal** program with respect to that property in analogy with our definition of FC-normal NRS.

Definition 5.2 *(a) Let P be a general program, let Con be a consistency property with respect to P. Call P* **FC-normal with respect to** *Con if for every clause $C = p \leftarrow q_1, \ldots, q_n, \neg r_1, \ldots, \neg r_m$ such that $C \in ground(P) - ground(H(P))$, and for every consistent fixpoint A of $T_{H(P)}$, if $q_1, \ldots, q_n \in A$, $p, r_1, \ldots, r_m \notin A$ we have:*
(1) $A \cup \{p\} \in Con$
(2) $A \cup \{p, r_i\} \notin Con$ for all $1 \leq i \leq m$.
(b) P is called **FC-normal** *if there exists a consistency property Con such that P is FC-normal with respect to Con.*

Next we translate Example 4.1 into the language of general logic programs.

Example 5.1 Let the Herbrand base consist of atoms a, b, c, d, e, f. Let the consistency property be defined by the following condition:
$A \notin Con$ if and only if either $\{c, d\} \subseteq A$ or $\{e, f\} \subseteq A$.

Now consider this program:

1) $a \leftarrow$
2) $b \leftarrow c$
3) $c \leftarrow b$
4) $c \leftarrow a, \neg d$
5) $e \leftarrow c, \neg f$

This program is FC-normal with respect to the consistency property described above and one can easily check that there is a unique stable model $M = \{a, b, c, e\}$.
If we add to this program the clause $f \leftarrow c, \neg e$, the resulting program is still FC-normal but now there are two stable models, $M_1 = \{a, b, c, e\}$ and $M_2 = \{a, b, c, f\}$.

We now have the following analogues of Theorem 4.1 and Theorem 4.2.

Theorem 5.3 *If P is an FC-normal program, then P possesses a stable model.*

Theorem 5.4 *If P is an FC-normal program with respect to the consistency property Con and $I \in Con$, then P possesses a stable model I' such that $I \subseteq I'$.*

The analogue of our forward chaining construction of extensions of an FC-normal NRS becomes the following. Let $\prec$ be a well-ordering of $ground(P) - ground(H(P))$. That is, the well-ordering $\prec$ determines some listing of the clauses of $ground(P) - ground(H(P))$, $\{c_\alpha : \alpha \in \gamma\}$ where γ is some ordinal. Let Θ_γ be the least cardinal such that $\gamma \leq \Theta_\gamma$. Our forward chaining construction will define an increasing sequence of subsets of $\mathcal{H}_P$, $\{T^\prec_\alpha\}_{\alpha \in \Theta_\gamma}$. This given, we will then define $T^\prec = \bigcup_{\alpha \in \Theta_\gamma} T^\prec_\alpha$ and show that $T^\prec$ is always an stable model of P. Moreover we shall show that all stable models of P arise from this construction.

The forward chaining construction of $T^\prec$.

<u>Case 0</u>. $T^\prec_0 = T_{H(P)} \Uparrow \omega(\emptyset)$

<u>Case 1</u>. Suppose that $\alpha = \eta + 1$ is a successor ordinal. Given $T^\prec_\eta$, let $\ell(\alpha)$ be the least $\lambda \in \gamma$ such that $c_\lambda = \varphi \leftarrow \alpha_1, \ldots, \alpha_n, \neg\beta_1, \ldots, \neg\beta_m$ where $\alpha_1, \ldots, \alpha_n \in T^\prec_\eta$ and $\beta_1, \ldots, \beta_m, \varphi \notin T^\prec_\eta$. If there is no such $\ell(\alpha)$, let $T^\prec_\alpha = T^\prec_\eta$. Otherwise let

$$T^\prec_\alpha = T_{H(P)} \Uparrow \omega(T^\prec_\eta \cup \{p_{\ell(\alpha)}\})$$

where $p_{\ell(\alpha)}$ is the head of $c_{\ell(\alpha)}$.

<u>Case 2</u>. α is a limit ordinal. Then let $T^\prec_\alpha = \bigcup_{\beta \in \alpha} T^\prec_\beta$.

We then get:

Theorem 5.5 *If P is an FC-normal program and $\prec$ is* any *well-ordering of $ground(P) - ground(H(P))$, then :*
(1) $T^{\prec}$ is a stable model of P.
(2) (completeness of the construction). Every stable model model of P is of the form $T^{\prec}$ for a suitably chosen ordering $\prec$ of $ground(P) - ground(H(P))$.

Example 5.2 If we consider the final extended program of Example 5.1, it is easy to check that any ordering $\prec_1$ in which the clause $C_1 = e \leftarrow c, \neg f$ precedes the clause $C_2 = f \leftarrow c, \neg e$ will have $T^{\prec_1} = M_1$ while any ordering $\prec_2$ in which C_2 precedes C_1 will have $T^{\prec_2} = M_2$.

Once again, we note that if we restrict ourselves to countable programs P, then we can restrict ourselves to orderings of order type ω. That is, suppose we fix some well-ordering $\prec$ of $ground(P) - ground(H(P))$ of order type ω. Thus, the well-ordering $\prec$ determines some listing of the clauses of $ground(P) - ground(H(P))$,$\{c_n : n \in \omega\}$. Again in this case, our forward chaining construction can be presented in a more straightforward manner. Our construction will define an increasing sequence of sets $\{T_n^{\prec}\}_{n\in\omega}$ in stages. This given, we will then define $T^{\prec} = \bigcup_{n\in\omega} T_n^{\prec}$.

The countable forward chaining construction of $T^{\prec}$.

Stage 0. Let $T_0^{\prec} = T_{H(P)} \Uparrow \omega(\emptyset)$.

Stage $n+1$. Let $\ell(n+1)$ be the least $s \in \omega$ such that $c_s = \varphi \leftarrow \alpha_1, \ldots, \alpha_k, \neg\beta_1, \ldots, \neg\beta_m$ where $\alpha_1, \ldots, \alpha_k \in T_n^{\prec}$ and $\beta_1, \ldots, \beta_m, \varphi \notin T_n^{\prec}$. If there is no such $\ell(n+1)$, let $T_{n+1}^{\prec} = T_n^{\prec}$. Otherwise let

$$T_{n+1}^{\prec} = T_{H(P)} \Uparrow \omega(T_n^{\prec} \cup \{p_{\ell(n+1)}\})$$

where $p_{\ell(n+1)}$ is the head of $c_{\ell(n+1)}$.

Theorem 5.6 *If P is a countable FC-normal program, and $\prec$ is* any *well-ordering of $ground(P) - ground(H(P))$ of order type ω, then :*
(1) $T^{\prec}$ is a stable model of P where $T^{\prec}$ is constructed via the countable forward chaining algorithm.
(2) (completeness of the construction). Every stable model model of P is of the form $T^{\prec}$ for a suitably chosen ordering $\prec$ of $ground(P) - ground(H(P))$ of order type ω where $T^{\prec}$ is constructed via the countable forward chaining algorithm.

Theorem 5.7 *If P is an FC-normal logic program with respect to Con, then every stable model M of P is in Con.*

FC-normal programs possess the "semi-monotonicity" property.

Theorem 5.8 *Let P_1, P_2 be two general programs such that $P_1 \subseteq P_2$ but $H(P_1) = H(P_2)$. Assume, in addition, that both are FC-normal with respect to the same consistency property. Then for every stable model M_1 of P_1, there is a stable model M_2 of P_2 such that*

1. *$M_1 \subseteq M_2$ and*
2. *$NG(M_1, P_1) \subseteq NG(M_2, P_2)$.*

Here given a logic program P and a stable model M, we let $NG(M, P)$ equal the set of all clauses $c = \varphi \leftarrow \alpha_1, \ldots, \alpha_k, \neg\beta_1, \ldots, \neg\beta_m$ in *ground*(P) such that $\alpha_1, \ldots, \alpha_k \in M$ and $\beta_1, \ldots, \beta_m \notin M$.

This is a very useful result. That is, if we consider our paradigm of the robot who moves are determined by a hardwired logic program as described in the introduction, this results tells us that if the robot is operating with respect to certain belief set or point of view and subsequently new clauses, FC-normal with respect to the consistency property ruling the behavior of the robot, are added, then the beliefs at that point do not need to be recomputed, just new point of view extending the current point of view can be formed. Adding new *Horn* rules, however, may require backtracking. The reason is that we may have chosen a belief explicitly contradicting these new facts.

The analogues of the remaining theorems of Section 4, also hold for FC-normal logic programs.

Theorem 5.9 *Let P be an FC-normal logic program with respect to a consistency property Con. Then if E_1 and E_2 are two distinct stable models of P, then $E_1 \cup E_2 \notin Con$.*

Theorem 5.10 *Let P be an FC-normal program with respect to a consistency property Con. Suppose that $T \Uparrow \omega(\{head(c) : c \in ground(P) - ground(H(P))\})$ is in Con where for any clause c, $head(c)$ denotes the head of the clause. Then P has a unique stable model.*

Theorem 5.11 *Suppose P is an FC-normal logic program and that $D \subseteq ground(P) - ground(H(P))$. Suppose further that E'_1 and E'_2 are distinct stable models of the program of $ground(P) \cup D$. Then P has distinct stable models E_1 and E_2 such that $E'_1 \subseteq E_1$ and $E'_2 \subseteq E_2$.*

To state the analogue of Theorem 7, we must define the notion of proof scheme for a logic program P. A *proof scheme* for p with respect to P is a sequence of triples $< \langle p_l, C_l, S_l \rangle >_{1 \leq l \leq n}$, with n a natural number, such that the following conditions all hold.

1. Each p_l is in $\mathcal{H}_P$. Each C_l is in $ground(P)$. Each S_l is a finite subset of $\mathcal{H}_P$.
2. p_n is p.
3. The S_l, C_l satisfy the following conditions. For all $1 \leq l \leq n$, one of **(a)**, **(b)**, **(c)** below holds.

   (a) C_l is $p_l \leftarrow$, and S_l is S_{l-1},

   (b) C_l is $p_l \leftarrow \neg s_1, \ldots, \neg s_r$ and S_l is $S_{l-1} \cup \{s_1, \ldots, s_r\}$, or

   (c) C_l is $p_l \leftarrow p_{m_1}, \ldots, p_{m_k}, \neg s_1, \ldots, \neg s_r$, $m_1 < l, \ldots, m_k < l$, and S_l is $S_{l-1} \cup \{s_1, \ldots, s_r\}$.

(We put $S_0 = \emptyset$).

Suppose that $\varphi = < \langle p_l, C_l, S_l \rangle >_{1 \leq l \leq n}$ is a proof scheme. Then $cln(\varphi)$ denotes the atom p_n and is called the *conclusion* of φ. Also, $supp(\varphi)$ is the set S_n and is called the *support* of φ.

Now suppose that P is FC-normal logic program with respect to the consistency property Con. Then we say a proof scheme $< \langle p_l, C_l, S_l \rangle >_{1 \leq l \leq n}$ is consistent with respect to Con if $\{p_1, \ldots p_n\} \in Con$. We then have the following.

Theorem 5.12 *Let P be an FC-normal logic program with respect to a consistency property Con. Then $\varphi \in \mathcal{H}_P$ is an element of some stable model of P if and only if φ has a consistent proof scheme with respect to Con.*

5.2 Default logic

In this subsection, we shall translate our results of Section 4 back into the language of default logic. We shall start with the translation between default logic and nonmonotonic rule systems as described in [MNR90] and [MNR92c]. Let U be the collection of all formulas of propositional logic $\mathcal{L}$. Recall a default theory $\langle D, W \rangle$ as a pair where D is a collection of default rules, that is, rules of form

$$\frac{\alpha \colon M\beta_1, \ldots, M\beta_m}{\psi}, \tag{10}$$

(where $\alpha, \beta_1, \ldots, \beta_m$, and ψ are formulas) and W a collection of formulas of the language $\mathcal{L}$.

Represent such a default theory as a rule system consisting of three lists:
(i) Elements $\gamma \in W$ are represented as rules:

$$\frac{:}{\gamma}$$

(ii) Rules of form (10) are represented as

$$\frac{\alpha\colon \neg\beta_1, \ldots, \neg\beta_m}{\gamma}$$

(That is, the restraints of the rule representing a default rule r have an additional negation in front).
(iii) Processing rules of logic. That is, all the monotonic rules of the system of classical logic.

We then have the following proposition from [MNR90]:

Proposition 5.13 *A collection $S \subseteq U$ is an extension of a system consisting of rules of types* (i), (ii), *and* (iii) *if and only if S is a default extension of $\langle D, W\rangle$.*

Given a default rule r as in (10), we let $prem(r) = \{\alpha\}$, $cons(r) = \{\neg\beta_1, \ldots, \neg\beta_m\}$, and $cln(r) = \psi$. If $m = 0$, then we say that r is a monotonic rule and otherwise we will say that r is a nonmonotonic rule. We let $mon(\langle D, W\rangle)$ denote the set of monotonic rules of $\langle D, W\rangle$ and $nmon(\langle D, W\rangle)$ denote the set of monotonic rules of $\langle D, W\rangle$. We say that a subset $S \subseteq \mathcal{S}$ is *monotonically closed* relative to $\langle D, W\rangle$, if $W \subset S$, $Cn(S) = S$ and for any montonic rule $\frac{\alpha:}{\psi}$ in $\langle D, W\rangle$, it is the case that ψinS if $\alpha \in S$. Thus a set S containing W is montonically closed relative to $\langle D, W\rangle$, if S is closed under the application of all monotonic rules of $\langle D, W\rangle$ as well as being closed under logical consequence. It is easy to see that the intersection of any two monotonically closed sets relative to $\langle D, W\rangle$ is also monotonically closed relative to $\langle D, W\rangle$ so that for any set $T \subseteq \mathcal{L}$, there is a smallest set S which contains T and is monotonically closed. We let $cl_{mon}(T)$ denote the smallest monotonically closed set relative to $\langle D, W\rangle$ which contains T.

Call a family of subsets of $\mathcal{L}$, Con a *consistency property* for $\langle D, W\rangle$ if it satisfies the following conditions:

1. $\emptyset \in Con$.
2. If $A \subseteq B$ and $B \in Con$ then $A \in Con$.
3. Con is closed under directed unions.
4. If $A \in Con$ then $cl_{mon}(A) \in Con$.

Given a consistency property Con, we say that a default rule rule as in (10) is **FC-normal with respect to** Con if for any theory $A \in Con$ such that $\alpha \in A$ and $\psi, \neg\beta_1, \ldots, \neg\beta_m$, are not in A, then $Cn(A \cup \{\psi\}) \in Con$ but $Cn(A \cup \{\psi, \neg\beta_i\}) \notin Con$ for any i. Then we say that a default theory $\langle D, W\rangle$ is **FC-normal** with respect to Con if each rule $r \in D$ is FC-normal with respect to Con.

Our next result will show that our definition of FC-normal default theories actually extends Reiter's original definition of normal default theories.

Theorem 5.14 *Every normal default theory $\langle D, W\rangle$ is an FC-normal default theory.*

Proof. First observe that since every rule in a normal default theory is of the form $\frac{\alpha : M\beta}{\beta}$, there are no monotonic rules in $\langle D, W\rangle$. Thus a set S is monotonically closed relative to $\langle D, W\rangle$ if and only if $Cn(S \cup W) = S$ (that is $W \subseteq S$ and $Cn(S) = S$).

There are two cases. First, suppose that W is a logically consistent set of formulas. In this case, it is easy to see that the set of logically consistent subsets S of the language $\mathcal{L}$ such that $W \cup S$ is logically consistent is a consistency property for $\langle D, W\rangle$. We note that if Con is just the set of logically consistent sets S in $\mathcal{L}$ such that $S \cup W$ is logically consistent, then every rule of the form $\frac{\alpha : M\beta}{\beta}$ is certainly FC-normal. That is, no consistent set can contain both β and $\neg\beta$ and if A is a consistent theory which contains W such that neither $\beta, \neg\beta$ are in A, then $Th(A \cup \{\beta\})$ is also consistent. Thus if W is logically consistent set, then $\langle D, W\rangle$ is FC-normal relative to the consistency property consisting of all logically consistent sets S such that $S \cup W$ is logically consistent. Second, if W is not logically consistent, then it is easy to see that for any set $S \subset \mathcal{L}$, $cl_{mon}(S) = Cn(S \cup W) = \mathcal{L}$. Thus in this case, the only possible consistency property with respect to $\langle D, W\rangle$ is the set of all subsets of $\mathcal{L}$. Moreover, the only monotonically closed set is $\mathcal{L}$. But then every rule of the form $\frac{\alpha : M\beta}{\beta}$ is FC-normal since there is no monotonically closed set T such that both β and $\neg\beta$ are not in T. Thus if W is logically inconsistent, then $\langle D, W\rangle$ is FC-normal with respect to the consistency property consisting of all subsets of $\mathcal{L}$. $\square$

Of course, it is easy to see that there are many FC-normal default theories which are not normal default theories since in FC-normal default theories we allow monotonic rules and we do not restrict nonmonotonic rules to be of the form $\frac{\alpha : M\beta}{\beta}$.

We then have the following analogues of the results of Section 4.

Theorem 5.15 *Let $\langle D, W\rangle$ be an FC-normal default theory with respect to consistency property Con, then there exists an extension of $\langle D, W\rangle$.*

Theorem 5.16 *Let $\langle D, W\rangle$ be an FC-normal default theory with respect to consistency property Con and let $I \in Con$. Then there exists an extension I' of $\langle D, W\rangle$ such that $I \subseteq I'$.*

The analogue of the forward chaining construction for FC-normal default theories is the following. Given an FC-normal default theory $\langle D, W\rangle$, fix some well-ordering $\prec$ of D. That is, the well-ordering $\prec$ determines some listing of the rules of D,$\{r_\alpha : \alpha \in \gamma\}$ where γ is some ordinal. Let Θ_γ be the least cardinal such that $\gamma \leq \Theta_\gamma$. In what follows, we shall assume that the ordering among ordinals is given by $\in$. Our forward chaining construction will define an increasing sequence of sets $\{E^\prec_\alpha\}_{\alpha \in \Theta_\gamma}$.

The forward chaining construction of $E^\prec$.

Case 0. Let $E^\prec_0 = cl_{mon}(W)$.

Case 1. $\alpha = \eta + 1$ is a successor ordinal.
Given $E^\prec_\eta$, let $\ell(\alpha)$ be the least $\lambda \in \gamma$ such that $r_\lambda = \frac{\alpha_1,\ldots,\alpha_p : M\beta_k,\ldots M\beta_k}{\psi}$ where $\alpha_1, \ldots, \alpha_p \in E^\prec_\eta$ and $\neg\beta_1, \ldots, \neg\beta_k,\ \psi \notin E^\prec_\eta$. If there is no such $\ell(\alpha)$, then let $E^\prec_{\eta+1} = E^\prec_\alpha = E^\prec_\eta$. Otherwise, let

$$E^\prec_{\eta+1} = E^\prec_\alpha = cl_{mon}(E^\prec_\eta \cup \{cln(r_{\ell(\alpha)})\}).$$

Case 2. α is a limit ordinal. Then let $E^\prec_\alpha = \bigcup_{\beta \in \alpha} E^\prec_\beta$.

This given, we have the following.

Theorem 5.17 *If $\langle D, W\rangle$ is an FC-normal default theory and $\prec$ is* any *well-ordering of D, then*

1. *$E^\prec$ is an extension of $\langle D, W\rangle$.*

2. *(completeness of the construction). Every extension of $\langle D, W\rangle$ is of the form $E^\prec$ for a suitably chosen well-ordering $\prec$ of D.*

Corollary 5.18 *Let $\langle D, W\rangle$ be an FC-normal default theory with respect to consistency property Con, then every extension of $\langle D, W\rangle$ is in Con.*

If we restrict ourselves to countable default theories $\langle D, W\rangle$, i.e. if the underlying propositional language countable, then we can restrict ourselves to orderings of order type ω where ω is the order type of the natural numbers. That is, suppose we fix some well-ordering $\prec$ of D of order type ω. Thus, the well-ordering $\prec$ determines some listing of the rules of D, $\{r_n : n \in \omega\}$. Our forward chaining construction can be presented in a more straightforward manner in this case. Our construction again will define an increasing sequence of sets $\{E^\prec_n\}_{n \in \omega}$ in stages.

The countable forward chaining construction of $E^{\prec}$.

Stage 0. Let $E_0^{\prec} = cl_{mon}(W)$.

Stage $n+1$. Let $\ell(n+1)$ be the least $s \in \omega$ such that $r_s = \frac{\alpha_1,\ldots,\alpha_p : M\beta_k,\ldots M\beta_k}{\psi}$ where $\alpha_1, \ldots, \alpha_p \in E_n^{\prec}$ and $\neg\beta_1, \ldots, \neg\beta_k,\ \psi \notin E_n^{\prec}$. If there is no such $\ell(n+1)$, then let $E_{n+1}^{\prec} = E_n^{\prec}$. Otherwise, let

$$E_{n+1}^{\prec} = E_n^{\prec} = cl_{mon}(E_n^{\prec} \cup \{cln(r_{\ell(n+1)})\}).$$

We then define $E^{\prec} = \bigcup_{n\in\omega} E_n^{\prec}$.

This given, we then have the following.

Theorem 5.19 *If $\langle D, W\rangle$ is a countable FC-normal default theory then*

1. *$E^{\prec}$ is an extension of $\langle D, W\rangle$ where $E^{\prec}$ is constructed via the countable forward chaining algorithm with respect to $\prec$, where $\prec$ is* any *well-ordering of D of order type ω.*

2. *(completeness of the construction.) Every extension of $\langle D, W\rangle$ is of the form $E^{\prec}$ for a suitably chosen well ordering $\prec$ of D of order type ω, where $E^{\prec}$ is constructed via the countable forward chaining algorithm.*

The "Semi-monotonicity" property holds for FC-normal default theories.

Theorem 5.20 *Let $\Delta_1 = \langle D_1, W\rangle$ and $\Delta_2 = \langle D_2, W\rangle$ be two FC-normal default theories with respect to a consistency property Con such that $mon(\Delta_1) = mon(\Delta_2)$ and $nmon(\Delta_1) \subseteq nmon(\Delta_2)$. Then for every extension E_1 of $\langle D_1, W\rangle$, there is an extension E_2 of $\langle D_2, W\rangle$ such that*

1. *$E_1 \subseteq E_2$ and*

2. *$GD(E_1, \Delta_1) \subseteq GD(E_2, \Delta_2)$.*

FC-normal default theories also satisfy the orthogonality of extension property with respect to their consistency property.

Theorem 5.21 *Let $\langle D, W\rangle$ be an FC-normal default theory with respect to the consistency property Con. Then if E_1 and E_2 are two distinct extensions of $\langle D, W\rangle$, then $E_1 \cup E_2 \notin Con$.*

Similarly we also have the following analogues of Theorems 2.4 and 2.5.

Theorem 5.22 *Let $\langle D, W\rangle$ be an FC-normal default theory with respect to a consistency property Con. Suppose that $W \cup \{cln(r) : r \in D\}$ is in Con. Then $\langle D, W\rangle$ has a unique extension.*

Theorem 5.23 *Let $\langle D, W\rangle$ be an FC-normal default theory and suppose that D' is a subset of D which contains $mon(\langle D, W\rangle)$. Suppose further that E_1' and E_2' are distinct extensions of $\langle D', W\rangle$. Then $\langle D, W\rangle$ has distinct extensions E_1 and E_2 such that $E_1' \subseteq E_1$ and $E_2' \subseteq E_2$.*

Finally there is also an analogue of Theorem 4.9. First we assume that the underlying logic has some proof systems consisting of finitely many axiom schema and finitely many rules $\theta_1, \ldots, \theta_k$. An (annotated) proof scheme for φ is a finite sequence

$$p = << \varphi_0, r_0, G_0 >, \ldots, < \varphi_m, r_m, G_m >> \tag{11}$$

such that $\varphi_m = \varphi$ and
(1) If $m = 0$ then:
(a) φ_0 is an instance of an axiom schema for $\mathcal{L}$ or $\varphi_0 \in W$, $r_0 = \varphi_0$, and $G_0 = \emptyset$, or
(b) φ is a conclusion of a rule $r = \frac{:M\beta_1,\ldots,M\beta_r}{\varphi}$, $r_0 = r$, and $G_0 = cons(r)$.
(2) If $m > 0$, $<< \varphi_i, r_i, G_i >>_{i=0}^{m-1}$ is a proof scheme of length m and either
(a) φ_m is a conclusion of $r = \frac{\varphi_i:M\beta_1,\ldots,m\beta_r}{\varphi_m}$ where $i < m$, $r_m = r$, and $G_m = G_{m-1} \cup cons(r)$ or
(b) there is some $i_0 < \ldots < i_s < m$ such that φ_m follows is the results of applying one of the rules of proof θ_j to $\alpha_{i_0}, \ldots, \alpha_{i_s}$, $r_m = \theta_j$, and $G_m = G_{m-1}$.

The formula φ_m is called the *conclusion* of p and is written $cln(p)$. The set G_m is called the *support* of p and is written $supp(p)$. We say that $\varphi \in \mathcal{L}$ has a consistent proof scheme with respect to a consistency property Con over $\langle D, W\rangle$ if and only if there is a proof scheme

$$p = << \varphi_0, r_0, G_0 >, \ldots, < \varphi_m, r_m, G_m >>$$

such that $\varphi_m = \varphi$ and $\{\varphi_0, \ldots, \varphi_m\} \in Con$. We then have the following.

Theorem 5.24 *Let $\langle D, W\rangle$ be an FC-normal default theory with respect to a consistency property Con. Then φ is an element of some extension of $\langle D, W\rangle$ if and only if φ has a consistent proof scheme with respect to Con.*

5.3 Logic programming with classical negation

We now discuss the so-called "logic programming with classical negation" of [GL90] as a chapter in the theory of nonmonotonic rule systems.
Recall the basic notions introduced in [GL90]. The collection of objects appearing in heads or bodies of clauses is the set of all literals, that is, atoms or negated atoms. In particular, a negated atom may appear in the head of a clause. Consider first "general Horn" clauses in which literals may appear in arbitrary places. To each set P of such clauses assign its *answer set*, the least collection A of literals satisfying the following two conditions:
(1) If $a \leftarrow b_1, \ldots, b_m$ is in P and $b_1, \ldots, b_m \in A$ then $a \in A$.
(2) If for some atom p, p and $\neg p$ are both in A, then A is the whole collection Lit of all literals.

Introduce a collection Str of **structural processing** rules over the set $U = Lit$. These are all monotone rules of the form:

$$\frac{p, \neg p:}{a}$$

for all atoms p and literals a.
Translate the clause: $a \leftarrow b_1, \ldots, b_n$ as rule:

$$\frac{b_1, \ldots, b_n:}{a}$$

and let tr(P) be the collection of translations of clauses in P plus the structural rules Str. Then we have

Proposition 5.25 *A subset $A \subseteq Lit$ is an answer set for P if and only if A is an extension of $tr(P)$. Since $tr(P)$ is a set of monotonic rules, such an answer set is the least fixpoint of the (monotonic) operator associated with the translation.*

Gelfond and Lifschitz then introduce general rules. Since the negation used in literals is not the "negation-as-failure" of general logic programming, Gelfond and Lifschitz introduce another negation symbol "*not*" and a general logic clause with classical negation in the form:

$$a \leftarrow b_1, \ldots, b_n, not(c_1), \ldots, not(c_m)$$

Then the **answer set** for a set P of clauses of this form is introduced by merging the operational procedure for the construction of stable models for a program (as introduced in [GL88]) with the procedure above. They define the answer set for a program with classical negation as follows:

Let $M \subseteq Lit$ and P be a general program. Define P/M as a collection of clauses lacking *not* obtained as follows:
(1) If a clause C contains a substring $not(a)$ where $a \in M$, then eliminate C altogether.
(2) In remaining clauses eliminate all substrings of the form $not(a)$.
The resulting program P/M lacks the symbol *not*, so the answer set is well defined. Let M' be the answer set for P/M. We call M an answer set for P precisely when $M' = M$.

Gelfond and Lifschitz give a computational procedure for finding such answer sets, and subsequently reduce computing them to computing default logic extensions. Here we give a general result showing that the construction of Gelfond and Lifschitz is faithfully represented within nonmonotonic rule systems; here is how. Define U to be Lit, and translate the clause:

$$a \leftarrow b_1, \ldots, b_n, not(c_1), \ldots, not(c_m)$$

as the rule:

$$\frac{b_1, \ldots, b_n : c_1, \ldots, c_m}{a}$$

The translation of the program P then consists of the translations of individual clauses C of P, incremented by the structural rules Str. We get the following result:

Proposition 5.26 *Let P be a general logic program with classical negation and N_P be the translation described above. Then a collection M is an answer set for P if and only if M is an extension for the rule system $\langle U, N_P \rangle$.*

Let $Lit_P = \mathcal{H}_P \cup \{\neg p : p \in \mathcal{H}_P\}$. It is easy to see that Lit_P plays the role of the universe U and $ground(P) \cup Str$ plays the role of the set of rules N in Section 4. Moreover, the *not*-free part of $ground(P)$ together with the structural rules Str, plays the role of the monotonic part of the rules N. Define $mon(P)$ to be the the *not*-free part of P incremented by Str. The immediate provability operator, $T_P(M)$, associated with $mon(P)$ is monotonic. Moreover, if the input M contains a pair of contradictory literals then $T_P(M)$ is the whole set Lit_P.

In this setting the notion of a consistency property becomes the following. Call a family of subsets of Lit_P, Con a **consistency property** over P if it satisfies the following conditions:

1. $\emptyset \in Con$.
2. If $A \subseteq B$ and $B \in Con$ then $A \in Con$.
3. Con is closed under directed unions.

4. If $A \in Con$ then $A \cup T_{mon(P)}(A) \in Con$.

5. $Lit_P \notin Con$

We have, as before,

Proposition 5.27 *If Con is a consistency property with respect to P and $A \in Con$, then $T_{mon(P)} \Uparrow \omega(A) \in Con$.*

Here for a general logic program Q, $T_{mon(Q)} \Uparrow \omega(A)$ is the cumulative fixed point of T_Q over A.

Given a consistency property, we define the concept of an **FC-normal** CN logic program with respect to that property in analogy with our definition of FC-normal NRS.

Definition 5.28 *(a) Let P be a* CN *logic program, let Con be a consistency property with respect to P. Call P* **FC-normal with respect to** *Con if for every clause $C = p \leftarrow q_1, \ldots, q_n, not(r_1), \ldots, not(r_m)$ such that $C \in ground(P) - ground(mon(P))$, and for every consistent fixpoint A of $T_{mon(P)}$, if $q_1, \ldots, q_n \in A$, $p, r_1, \ldots, r_m \notin A$ we have:*
(1) $A \cup \{p\} \in Con$
(2) $A \cup \{p, r_i\} \notin Con$ for all $1 \leq i \leq m$.
(b) P is called **FC-normal** *if there exists a consistency property Con such that P is FC-normal with respect to Con.*

Our condition (5) implies that there is a weakest consistency property. This is the consistency property based on the the absence of pair of complementary literals.

We now have the following analogues of Theorem 4.1 and Theorem 4.2.

Theorem 5.29 *If P is an FC-normal* CN *logic program, then P possesses an answer set.*

Theorem 5.30 *If P is an FC-normal* CN *logic program with respect to the consistency property Con and $I \in Con$, then P possesses an answer set I' such that $I \subseteq I'$.*

The analogue of our forward chaining construction of extensions of an FC-normal NRS becomes the following. Let $\prec$ be a well-ordering of $ground(P) - ground(mon(P))$. That is, the well-ordering $\prec$ determines some listing of the clauses of $ground(P) - ground(mon(P))$, $\{c_\alpha : \alpha \in \gamma\}$ where γ is some ordinal. Let Θ_γ be the least cardinal such that $\gamma \leq \Theta_\gamma$. Our forward chaining construction will define an increasing sequence of subsets of Lit_P, $\{T^{\prec}_\alpha\}_{\alpha\in\Theta_\gamma}$. This given, we will then define $T^{\prec} = \bigcup_{\alpha\in\Theta_\gamma} T^{\prec}_\alpha$ and show that $T^{\prec}$ is always an answer set for P. Moreover we shall show that all answer sets for P arise from this construction.

The forward chaining construction of $T^{\prec}$.

Case 0. $T^{\prec}_0 = T_{mon(P)} \Uparrow \omega(\emptyset)$

Case 1. Suppose that $\alpha = \eta+1$ is a successor ordinal. Given $T^{\prec}_\eta$, let $\ell(\alpha)$ be the least $\lambda \in \gamma$ such that $c_\lambda = \varphi \leftarrow \alpha_1, \ldots, \alpha_n, not(\beta_1), \ldots, not(\beta_m)$ where $\alpha_1, \ldots, \alpha_n \in T^{\prec}_\eta$ and $\beta_1, \ldots, \beta_m, \varphi \notin T^{\prec}_\eta$. If there is no such $\ell(\alpha)$, let $T^{\prec}_\alpha = T^{\prec}_\eta$. Otherwise let

$$T^{\prec}_\alpha = T_{mon(P)} \Uparrow \omega(T^{\prec}_\eta \cup \{p_{\ell(\alpha)}\})$$

where $p_{\ell(\alpha)}$ is the head of $c_{\ell(\alpha)}$.

Case 2. α is a limit ordinal. Then let $T^{\prec}_\alpha = \bigcup_{\beta\in\alpha} T^{\prec}_\beta$.

We then get:

Theorem 5.31 *If P is an FC-normal* CN *logic program and $\prec$ is* any *well-ordering of $ground(P) - ground(mon(P))$, then :*

1. *$T^{\prec}$ is an answer set for P.*

2. *(completeness of the construction). Every answer set for P is of the form $T^{\prec}$ for a suitably chosen ordering $\prec$ of $ground(P) - ground(mon(P))$.*

Once again, we note that if we restrict ourselves to countable programs P, then we can restrict ourselves to orderings of order type ω. That is, suppose we fix some well-ordering $\prec$ of $ground(P) - ground(mon(P))$ of order type ω. Thus, the well-ordering $\prec$ determines some listing of the clauses of $ground(P) - ground(mon(P))$,$\{c_n : n \in \omega\}$. Again in this case, our forward chaining construction can be presented in a more straightforward manner. Our construction will define an increasing sequence of sets $\{T^{\prec}_n\}_{n\in\omega}$ in stages. This given, we will then define $T^{\prec} = \bigcup_{n\in\omega} T^{\prec}_n$.

The countable forward chaining construction of $T^{\prec}$.

Stage 0. Let $T^{\prec}_0 = T_{mon(P)} \Uparrow \omega(\emptyset)$.

Stage $n+1$. Let $\ell(n+1)$ be the least natural number s such that $c_s = \varphi \leftarrow \alpha_1, \ldots, \alpha_k, not(\beta_1), \ldots, not(\beta_m)$ where $\alpha_1, \ldots, \alpha_k \in T_n^{\prec}$ and $\beta_1, \ldots, \beta_m, \varphi \notin T_n^{\prec}$. If there is no such $\ell(n+1)$, let $T_{n+1}^{\prec} = T_n^{\prec}$. Otherwise let

$$T_{n+1}^{\prec} = T_{H(P)} \Uparrow \omega(T_n^{\prec} \cup \{p_{\ell(n+1)}\})$$

where $p_{\ell(n+1)}$ is the head of $c_{\ell(n+1)}$.

Theorem 5.32 *If P is a countable FC-normal* CN *logic program, and $\prec$ is* any *well-ordering of $ground(P) - ground(mon(P))$ of order type ω, then :*
(1) $T^{\prec}$ is a answer set for P where $T^{\prec}$ is constructed via the countable forward chaining algorithm.
(2) (completeness of the construction). Every answer set for P is of the form $T^{\prec}$ for a suitably chosen ordering $\prec$ of $ground(P) - ground(mon(P))$ of order type ω where $T^{\prec}$ is constructed via the countable forward chaining algorithm.

Theorem 5.33 *If P is an FC-normal* CN *logic program with respect to Con, then every answer set model M for P is in Con.*

FC-normal programs possess "semi-monotonicity" property.

Theorem 5.34 *Let P_1, P_2 be two* CN *logic programs such that $P_1 \subseteq P_2$ but $mon(P_1) = mon(P_2)$. Assume, in addition, that both are FC-normal with respect to the same consistency property. Then for every answer set M_1 for P_1, there is an answer set M_2 for P_2 such that*

1. *$M_1 \subseteq M_2$ and*
2. *$NG(M_1, P_1) \subseteq NG(M_2, P_2)$.*

Here given a CN logic program P and an answer set M, we let $NG(M, P)$ equal the set of all clauses $c = \varphi \leftarrow \alpha_1, \ldots, \alpha_k, not(\beta_1), \ldots, not(\beta_m)$ in $ground(P)$ such that $\alpha_1, \ldots, \alpha_k \in M$ and $\beta_1, \ldots, \beta_m \notin M$.

The analogues of the remaining theorems of Section 4, also hold for FC-normal logic programs with classical negation.

Theorem 5.35 *Let P be an FC-normal* CN *logic program with respect to a consistency property Con. Then if E_1 and E_2 are two distinct answer sets for P, then $E_1 \cup E_2 \notin Con$.*

Theorem 5.36 *Let P be an FC-normal* CN *logic program with respect to a consistency property Con. Suppose that $T \Uparrow \omega(\{head(c) : c \in ground(P) - ground(mon(P))\})$ is in Con where for any clause c, $head(c)$ denotes the head of the clause. Then P has a unique answer set.*

Theorem 5.37 *Suppose P is an FC-normal* CN *logic program and that $D \subseteq ground(P) - ground(mon(P))$. Suppose further that E'_1 and E'_2 are distinct answer sets for the program of $ground(P) \cup D$. Then P has distinct answer sets E_1 and E_2 such that $E'_1 \subseteq E_1$ and $E'_2 \subseteq E_2$.*

To state the analogue of Theorem 7, we must define the notion of proof scheme for a logic program P. A *proof scheme* for p with respect to P is a sequence of triples $< \langle p_l, C_l, S_l \rangle >_{1 \leq l \leq n}$, with n a natural number, such that the following conditions all hold.

1. Each p_l is in Lit_P. Each C_l is in $ground(P)$. Each S_l is a finite subset of Lit_P.
2. p_n is p.
3. The S_l, C_l satisfy the following conditions. For all $1 \leq l \leq n$, one of **(a)**, **(b)**, **(c)** below holds.
   - **(a)** C_l is $p_l \leftarrow$, and S_l is S_{l-1},
   - **(b)** C_l is $p_l \leftarrow not(s_1), \ldots, not(s_r)$ and S_l is $S_{l-1} \cup \{s_1, \ldots, s_r\}$, or
   - **(c)** C_l is $p_l \leftarrow p_{m_1}, \ldots, p_{m_k}, not(s_1), \ldots, not(s_r)$, $m_1 < l, \ldots, m_k < l$, and S_l is $S_{l-1} \cup \{s_1, \ldots, s_r\}$.

(We put $S_0 = \emptyset$).

Suppose that $\varphi =< \langle p_l, C_l, S_l \rangle >_{1 \leq l \leq n}$ is a proof scheme. Then $cln(\varphi)$ denotes atom p_n and is called the *conclusion* of φ. Also, $supp(\varphi)$ is the set S_n and is called the *support* of φ.

Now suppose that P is an FC-normal logic program with respect to the consistency property Con. Then we say a proof scheme $< \langle p_l, C_l, S_l \rangle >_{1 \leq l \leq n}$ is consistent with respect to Con if $\{p_1, \ldots p_n\} \in Con$. We then have the following.

Theorem 5.38 *Let P be an FC-normal* CN *logic program with respect to a consistency property Con. Then $\varphi \in Lit_P$ is an element of some answer set for P if and only if φ has a consistent proof scheme with respect to Con.*

5.4 Truth Maintenance Systems

In this and the next section, we shall discuss two other nonmonotonic formalism which have appeared in the literature. In each of these cases, we shall be content to simply translate these formalisms into nonmonotonic rules systems and leave it to the reader to translate the definition of FC-normal and the statements of its various properties.

Our description takes care of both truth maintenance systems as defined by Doyle [Doy79] and De Kleer [dK86], with subsequent contributions of Reinfrank, Dressler, and Brewka [RDB89].

Let At be a collection of atoms. By a rule over At we mean a object of the form $r = \langle A \mid B \rangle \rightarrow c$ where $A, B \subseteq At$, $c \in At$. A truth maintenance system (TMS for short) is a collection of rules.

Let S be a TMS. Given $M \subseteq At$, an M-derivation of an atom $a \in At$ is a finite sequence $< a_1, \ldots, a_n >$ satisfying the conditions:
(1) $a_n = a$.
(2) For every $j \leq n$, either a rule $\langle \emptyset \mid \emptyset \rangle \rightarrow a_j$ belongs to S or
there is a rule $\langle A \mid B \rangle \rightarrow a_j$ in S such that $A \subseteq \{a_1, \ldots, a_{j-1}\}$, $B \cap M = \emptyset$.

We call M a TMS-extension of S if and only if M has the property that M consists of precisely these atoms that possess an M-derivation.

It is easy to reconstruct, in this setting, truth maintenance as logic programming. Namely, we can translate a rule $\langle A \mid B \rangle \rightarrow c$ as a program clause $c \leftarrow a_1, \ldots, a_m$, $not(b_1), \ldots, not(b_n)$ where $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_n\}$. In this fashion, TMS-extensions become stable models of the resulting programs. This also implies that one can also construct truth-maintenance systems with literals.

To do this properly one considers (as in the case rules which involve literals and also"built-in" rules $\langle \{a, \neg a\}, \emptyset \rangle \rightarrow c$. For more about these, see ([MT93]).

The results on FC-normal logic programs as well as on FC-normal logic programs with classical negation allow us to prove analogous results for TMS, and also TMS with literals. We shall not pursue this matter further in this paper.

5.5 McDermott and Doyle systems

McDermott and Doyle [MD80] and McDermott [McD82] investigated another system of nonmonotonic reasoning. This system is based on modal logic. Here is a short description of that approach and the description how it fits into our framework. Let $\mathcal{L}_L$ be the propositional modal language based on modal operator L (expressing the necessity operator). We consider a strong notion of proof based on the application of the necessitation rule to all formulas, not just all theorems, of the logic under consideration. That is, this notion of proof from a set of formulas I allows us to apply necessitation to all formulas previously proved.

Let $\mathcal{S}$ be a modal logic. Examples of such a logic includes the familiar $S4$, $S5$, K or even a logic that does not includes the schemes of K. We associate with $\mathcal{S}$ its consequence operation based on the above strong notion of proof. We denote it by $Cn_{\mathcal{S}}(\cdot)$. We now introduce the notion of **$\mathcal{S}$-expansion**. Given a set of formulas $I \subseteq \mathcal{L}_L$, we say that a theory $T \subseteq \mathcal{L}_L$ is a $\mathcal{S}$-expansion of I if

$$T = Cn_{\mathcal{S}}(I \cup \{\neg L\varphi \colon \varphi \notin T\}) \tag{12}$$

Notice that the role of the logic $\mathcal{S}$ here is slightly different than in the usual applications of modal logic. $\mathcal{S}$ serves as means of *reconstruction* of T from the initial assumptions I and the *negative introspection* with respect to T. It should be clear that regardless of what $\mathcal{S}$ is (it does not even need to be included in $S5$) that an expansion of any theory is closed under $S5$-consequence. It is the discipline of reconstruction that makes the difference. Note the weaker the logic, the more difficult it is to reconstruct.

We show now how this formalism can be faithfully represented as a nonmonotonic rule system. Let $\mathcal{S}$ be a fixed modal logic, axiomatized by a set of axioms AX. We define a rule system $S_{\mathcal{S}}$ as follows. The universe U of our system is $\mathcal{L}_L$. The set N consists of the following five groups of rules:

1. $\frac{:}{\varphi}$, where φ ranges over all the axioms of propositional logic in the language $\mathcal{L}_L$, treating every formula of the form $L\psi$ as an atom.
2. $\frac{:}{\varphi}$, where φ ranges over all the the axioms of the logic $\mathcal{S}$.
3. $\frac{\varphi:}{L\varphi}$ for all the formulas $\varphi \in \mathcal{L}_L$.
4. $\frac{\varphi, \varphi \supset \psi:}{\psi}$ for all the formulas $\varphi, \psi \in \mathcal{L}_L$.
5. $\frac{:\varphi}{\neg L\varphi}$ for all $\varphi \in \mathcal{L}_L$.

Notice that the groups (1), (2), (3), and (4) of rules are monotonic, only the group (5) consists of nonmonotonic rules.

We have the following result

Theorem 5.39 *Let $\mathcal{S}$ be a modal logic. Let $I \subseteq \mathcal{L}_L$. Then T is an $\mathcal{S}$-expansion of of I if end only if T is an extension of I in the nonmonotonic rule system $S_{\mathcal{S}}$.*

This result allows us to apply the theory developed in this paper to $\mathcal{S}$-expansions. That is, we can introduce the notion of consistency property for a modal logic (notice that we have just one system for any given modal logic). If the system $S_{\mathcal{S}}$ is FC-normal for such property then, in particular every set of formulas consistent with respect to that property possesses an extension.

6 The Complexity of Extensions for a Recursive FC-normal NRS.

This section will be divided into two subsections. First we shall introduce some preliminaries from recursion theory and various classes of recursive rule systems so that we can make our statements about the complexity of extensions for recursive FC-normal NRS's precise. We shall also provide a brief review of what is known about the complexity of extensions in recursive nonmonotonic rule systems. In the second subsection, we shall state our new results about the complexity of extensions in recursive FC-normal nonmonotonic rule systems. The proofs of these new results will be deferred until Section 8

6.1 Preliminaries

Let ω denote the set of natural numbers. The canonical index, $can(X)$, of finite set $X = \{x_1 < \ldots < x_n\} \subseteq \omega$ is defined as $2^{x_1} + \ldots + 2^{x_n}$ and the canonical index of $\emptyset$ is defined as 0. Let D_k be the finite set whose canonical index is k, i.e., $can(D_k) = k$.

We shall identify a rule r with a triple $< k, l, \varphi >$ where $D_k = prem(r)$, and $D_l = cons(r)$, $\varphi = c(r)$. In this way, when $U \subseteq \omega$ we can think about N as a subset of ω as well. This given, we then say that a NRS $\mathcal{S} = \langle U, N\rangle$ is **recursive** if U and N are recursive subsets of ω

Next we shall define various types of recursive trees and Π^0_1 classes. Let $[,]\colon \omega\times\omega \rightarrow \omega$ be a fixed one-to-one and onto recursive pairing function such that the projection functions π_1 and π_2 defined by $\pi_1([x, y]) = x$ and $\pi_2([x, y]) = y$ are also recursive. Extend our pairing function to code n-tuples for $n > 2$ by the usual inductive definition, that is, let $[x_1, \ldots, x_n] = [x_1, [x_2, \ldots, x_n]]$ for $n \geq 3$. Let $\omega^{<\omega}$ be the set of all finite sequences from ω, let $2^{<\omega}$ be the set of all finite sequences of 0's and 1's. Given $\alpha =< \alpha_1, \ldots, \alpha_n >$ and $\beta =< \beta_1, \ldots, \beta_k >$ in $\omega^{<\omega}$, write $\alpha \sqsubseteq \beta$ if α is initial segment of β, i.e. , if $n \leq k$ and $\alpha_i = \beta_i$ for $i \leq n$. In this paper, we identify each finite sequence $\alpha =< \alpha_1, \ldots, \alpha_n >$ with its code $c(\alpha) = [n, [\alpha_1, \ldots, \alpha_n]]$ in ω. Let 0 be the code of the empty sequence $\emptyset$. When we say that a set $S \subseteq \omega^{<\omega}$ is recursive, recursively enumerable, etc., what we mean is that the set $\{c(\alpha)\colon \alpha \in S\}$ is recursive, recursively enumerable, etc. Define a **tree** T to be a nonempty subset of $\omega^{<\omega}$ such that T is closed under initial segments. Call a function $f\colon \omega \rightarrow \omega$ an infinite **path** through T provided that for all n, $< f(0), \ldots, f(n) >\in T$. Let $[T]$ be the set of all infinite paths through T. Call a set A of functions a Π^0_1-class if there exists a recursive predicate R such that $A = \{f\colon \omega \to \omega : \forall n(R([f(0), \ldots, f(n)])\}$. Call a Π^0_1-class A **recursively bounded** if there exists a recursive function $g\colon \omega \to \omega$ such that $\forall f \in A \forall_n (f(n) \leq g(n))$. It is not difficult to see that if A is a Π^0_1-class, then $A = [T]$ for some recursive tree $T \subseteq \omega^{<\omega}$. Say that a tree $T \subseteq \omega^{<\omega}$ is **highly recursive** if T is a recursive, finitely branching tree, and also there is a recursive procedure which, applied to $\alpha =< \alpha_1, \ldots, \alpha_n >$ in T, produces a canonical index of the set of immediate successors of α in T. Then if A is a recursively bounded Π^0_1-class, it is easy to show that $A = [T]$ for some highly recursive tree $T \subseteq \omega^{<\omega}$, see [JS72b]. For any set $A \subseteq \omega$, let $A' = \{e\colon \{e\}^A(e) \ \text{ is defined}\}$ be the jump of A, let $0'$ denote the jump of the empty set $\emptyset$. We write $A \leq_T B$ if A is Turing reducible to B and $A \equiv_T B$ if $A \leq_T B$ and $B \leq_T B$.

Formally, say that there is an effective, one-to-one degree preserving correspondence between the set of extensions $\mathcal{E}(\mathcal{S})$ of a recursive nonmonotonic rule system $\mathcal{S} = \langle U, N\rangle$ and the set of infinite paths $[T]$ through a recursive tree T if there are indices e_1 and e_2 of oracle Turing machines such that
(i) $\forall f \in [T]\{e_1\}^{gr(f)} = E_f \in \mathcal{E}(\mathcal{S})$,
(ii) $\forall E \in \mathcal{E}(\mathcal{S})\{e_2\}^E = f_E \in [T]$, and
(iii) $\forall f \in [T]\forall_{E\in\mathcal{E}(\mathcal{S})}(\{e_1\}^{gr(f)} = E$ if and only if $\{e_2\}^E = f)$.
where $\{e\}^B$ denotes the function computed by the e^{th} oracle machine with oracle B. Also, write $\{e\}^B = A$ for a set A if $\{e\}^B$ is a characteristic function of A, and for a function $f\colon \omega \to \omega$, $gr(f) = \{[x, f(x)]\colon x \in \omega\}$. Condition (i) says that the branches of the tree T uniformly produce extensions via an algorithm with index e_1. Condition (ii) says that extensions of $\mathcal{S}$ uniformly produce branches of the tree T via an algorithm with index e_2. Condition (iii) asserts that if $\{e_1\}^{gr(f)} = E_f$, then f is Turing equivalent to E_f. In the sequel we shall not explicitly construct the indices e_1 and e_2, but it will be clear that such indices can be constructed in each case.

There are two important subclasses of recursive NRS's introduced in [MNR92a], namely locally finite and highly recursive nonmonotonic rules systems. Say that the system $\langle U, N\rangle$ is **locally finite** if for each $\varphi \in U$, there exist only finitely many $\prec$-minimal proof schema with conclusion φ. If $\langle U, N\rangle$ is locally finite, then for every φ, there exists a finite set of derivations Dr_φ such that all the derivations of φ are inessential extensions of derivations in Dr_φ. That is, if p is a derivation of φ, then there is a derivation $p_1 \in Dr_\varphi$ such that $p_1 \prec p$. Finally, say that $\langle U, N\rangle$ is **highly recursive** if $\langle U, N\rangle$ is recursive, locally finite, and the map $\varphi \mapsto can(Dr_\varphi)$ is partial recursive. The latter means that there is an effective procedure which, when applied to any $\varphi \in U$, produces a canonical index of the set of all $\prec$-minimal proof schema with conclusion φ.

This given, we can now state some basic results from [MNR92c, MNR92b, MNR92a] on the complexity of extensions in recursive nonmonotonic rule systems.

Theorem 6.1 *For any highly recursive NRS system $\mathcal{S} = \langle U, N\rangle$, there is a highly recursive tree $T_{\mathcal{S}}$ such that there is an effective one-to-one degree preserving correspondence between $[T_{\mathcal{S}}]$ and $\mathcal{E}(\mathcal{S})$. Vice versa, for any highly recursive tree T, there is a highly recursive NRS system $\mathcal{S}_T = \langle U, N\rangle$ such that there is an effective one-to-one degree preserving correspondence between $[T]$ and $\mathcal{E}(\mathcal{S}_T)$.*

Theorem 6.2 *For any locally finite recursive NRS system $\mathcal{S} = \langle U, N\rangle$, there is a tree $T_{\mathcal{S}}$ which is highly recursive in $0'$ such that there is an effective one-to-one degree preserving correspondence between $[T_{\mathcal{S}}]$ and $\mathcal{E}(\mathcal{S})$. Vice versa, for any highly recursive tree T in $0'$, there is a locally finite recursive NRS system $\mathcal{S}_T = \langle U, N\rangle$ such that there is an effective one-to-one degree preserving correspondence between $[T]$ and $\mathcal{E}(\mathcal{S}_T)$.*

Theorem 6.3 *For any recursive NRS system $\mathcal{S} = \langle U, N\rangle$, there is a recursive tree $T_{\mathcal{S}}$ such that there is an effective one-to-one degree preserving correspondence between $[T_{\mathcal{S}}]$ and $\mathcal{E}(\mathcal{S})$. Vice versa, for any recursive tree T, there is a recursive NRS system $\mathcal{S}_T = \langle U, N\rangle$ such that there is an effective one-to-one degree preserving correspondence between $[T]$ and $\mathcal{E}(\mathcal{S}_T)$.*

Because the set of degrees of paths through trees have been extensively studied in the literature, we immediately can derive a number of corollaries about the degrees of extensions in recursive NRS systems. We shall give a few of these corollaries below. We begin with some consequences of Theorem 6.1. First there are some basic results which guarantee that there are extensions of a highly recursive NRS system which are not too complex. Let $\mathbf{0}$ denote the degree of recursive sets and $\mathbf{0}'$ its jump. Call A *low* if $A' \equiv_T 0'$. This means that A is called low provided that the jump of A is as small as possible with respect to Turing degrees. The following corollary is an immediate consequence of Theorem 4.1 and the work of Jockusch and Soare [JS72b].

Corollary 6.4 *Let $\mathcal{S} = \langle U, N\rangle$ be a highly recursive nonmonotonic rule system such that $\mathcal{E}(\mathcal{S}) \neq \emptyset$. Then*
(i) There exists an extension E of $\mathcal{S}$ such that E is low.
(ii) If $\mathcal{S}$ has only finitely many extensions, then every extension E of $\mathcal{S}$ is recursive.

In the other directions, there are a number of corollaries of the Theorem 4.1 which allow us to show that there are highly recursive NRS systems $\mathcal{S}$ such that the set of degrees realized by elements of $\mathcal{E}(\mathcal{S})$ are quite complex. Again all these corollaries follow by transferring results of Jockusch and Soare [JS72b, JS72a].

Corollary 6.5 *1. There is a highly recursive nonmonotonic rule system $\langle U, N\rangle$ such that $\langle U, N\rangle$ has $2^{\aleph_0}$ extensions but no recursive extensions.*

2. *There is a highly recursive nonmonotonic rule system $\langle U, N\rangle$ such that $\langle U, N\rangle$ has $2^{\aleph_0}$ extensions and any two extensions $E_1 \neq E_2$ of $\langle U, N\rangle$ are Turing incomparable.*

3. *If $\mathbf{a}$ is any Turing degree such that $\mathbf{0} <_T \mathbf{a} \leq_T \mathbf{0}'$, then there is a highly recursive nonmonotonic rule system $\langle U, N\rangle$ such that $\langle U, N\rangle$ has $2^{\aleph_0}$ extensions but no recursive extensions and $\langle U, N\rangle$ has an extension of degree $\mathbf{a}$.*

4. *If $\mathbf{a}$ is any Turing degree such that $\mathbf{0} <_T \mathbf{a} \leq_T \mathbf{0}'$, then there is a highly recursive nonmonotonic rule system $\langle U, N\rangle$ such that $\langle U, N\rangle$ has $\aleph_0$ extensions, $\langle U, N\rangle$ has an extension E of degree $\mathbf{a}$ and if $E' \neq E$ is an extension of $\langle U, N\rangle$, then E' is recursive.*

5. *There is a highly recursive nonmonotonic rule system $\langle U, N\rangle$ such that $\langle U, N\rangle$ has $2^{\aleph_0}$ extensions and if $\mathbf{a}$ is the degree of any extension E of $\langle U, N\rangle$ and $\mathbf{b}$ is any recursively enumerable degree such that $\mathbf{a} <_T \mathbf{b}$, then $\mathbf{b} \equiv_T \mathbf{0}'$.*

6. *If $\mathbf{a}$ is any recursively enumerable Turing degree, then there is a highly recursive nonmonotonic rule system $\langle U, N\rangle$ such that $\langle U, N\rangle$ has $2^{\aleph_0}$ extensions and the set of recursively enumerable degrees $\mathbf{b}$ which contain an extension of $\langle U, N\rangle$ is precisely the set of all recursively enumerable degrees $\mathbf{b} \geq_T \mathbf{a}$.*

Now the situation for locally finite recursive NRS rule systems is very similar to the situation for highly recursive NRS systems except that the degrees of extensions may be more complex. Essentially every result in Corollaries 4.4 and 4.5 hold for locally finite recursive NRS systems where each statement is taken relative to an $\mathbf{0}'$ oracle. See [MNR92c] for further details. However, the situation for general recursive NRS systems is quite different. That is, for general recursive NRS systems the set of elements of $\mathcal{E}(\mathcal{S})$ may be extremely complex. For example, all we can say in the positive direction is the following.

Corollary 6.6 1. *Every recursive NRS system $\mathcal{S} = \langle U, N\rangle$ which has an extension has an extension E such that $E \leq_T B$ where B is a complete Π^1_1-set.*

2. *If $\mathcal{S} = \langle U, N\rangle$ is a recursive NRS system with a unique extension E, then E is hyperarithmetic.*

In the opposite direction we have the following results, see[MNR92b].

Corollary 6.7 1. *There a recursive NRS system $\mathcal{S} = \langle U, N\rangle$ such that $\mathcal{S}$ has an extension but $\mathcal{S}$ has no extension which is hyperarithmetic.*

2. *For each recursive ordinal α, there exists a recursive NRS system $\mathcal{S} = \langle U, N\rangle$ possessing a unique extension E such that $E \equiv_T \mathbf{0}^{(\alpha)}$.*

6.2 Recursion theory of extensions of FC-normal NRS

In this subsection, we shall state our results on the complexity of extensions in a recursive FC-normal nonmonotonic rule systems. Our first result of this subsection will show that under the assumption of FC-normality that even recursive nonmonotonic rule systems are guaranteed to have at least one relatively well behaved extension which is in great contrast to Corollary 6.7.

Theorem 6.8 *Suppose that $\mathcal{S} = \langle U, N \rangle$ is a recursive nonmonotonic rule system and $\mathcal{S}$ is FC-normal. Then $\mathcal{S}$ has an extension E such that E is r.e. in $0'$ and hence $E \leq_T 0''$.*

We note that Theorem 6.8 is in some sense the best possible. That is, results from [MNR93a] show that the following holds. Given sets $A, B \subseteq \omega$, let $A \oplus B = \{2x : x \in A\} \cup \{2x+1 : x \in B\}$.

Theorem 6.9 *Let A be any r.e. set and B be any set which is r.e. in A, i.e. $B = \{x : \varphi_e(x) \downarrow\}$. Then there is a recursive FC-normal NRS $\mathcal{S} = \langle U, N \rangle$ such $\mathcal{S}$ has a unique extension E and $E \equiv_T A \oplus B$. In particular, if B is any set which is r.e. in $\mathbf{0}'$ and $B \geq_T \mathbf{0}'$, then there is an FC-normal NRS $\mathcal{S}$ such that $\mathcal{S}$ has a unique extension E and $E \equiv_T B$.*

However we shall show that if either $mon(\mathcal{S})$ or $nmon(\mathcal{S})$ is finite, then we can improve on Theorem 6.8. That is, we have the following.

Theorem 6.10 *Let $\mathcal{S} = \langle U, N \rangle$ be a recursive rule system such that $\mathcal{S}$ is FC-normal and $nmon(\mathcal{S})$ is finite, then every extension of $\mathcal{S}$ is r.e.*

We say that a recursive nonmonotonic rule system $\mathcal{S} = \langle U, N \rangle$ is **monotonically decidable** if the monotonic closure of any finite set F is recursive and there is a uniform effective procedure to go from a canonical index of a finite set F to a recursive index of the $cl_{mon}(F)$, i.e. if there is a recursive function f such that for all k, $\phi_{f(k)}$ is the characteristic function of $cl_{mon}(D_k)$. It is easy to see that if $mon(\mathcal{S})$ is finite, then the recursive nonmonotonic rule system $\mathcal{S} = \langle U, N \rangle$ is automatically monotonically decidable.

Theorem 6.11 *Let $\mathcal{S} = \langle U, N \rangle$ be a recursive nonmonotonic rule system such that $\mathcal{S}$ is FC-normal and monotonically decidable, then $\mathcal{S}$ has an extension which is r.e.*

Next we consider the case where $\mathcal{S} = \langle U, N \rangle$ is a finite FC-normal nonmonotonic rule system, i.e. we assume that both U and N are finite. In this case, we shall see that our forward chaining algorithm runs in polynomial time.

For complexity considerations, we shall assume that the elements of U are coded by strings over some finite alphabet Σ. Thus every $a \in U$ will have some length which we denote by $||a||$. Next, for a rule $r = \frac{a_1, \ldots, a_n : b_1, \ldots, b_m}{c}$, we define $||r|| = (\sum_{i \leq n} ||a_i||) + (\sum_{i \leq m} ||b_j||) + ||c||$. Finally, for a set Q of rules, we define

$$||Q|| = \sum_{r \in Q} ||r||.$$

Theorem 6.12 *Suppose $\mathcal{S} = \langle U, N \rangle$ is a finite FC-normal nonmonotonic rule system and $\prec$ is some well-ordering of $nmon(S)$. Then $E^{\prec}$ as constructed via our forward chaining algorithm can be computed in time $O(||mon(\mathcal{S})|| \cdot ||nmon(\mathcal{S})|| + ||nmon(\mathcal{S})||^2)$.*

We note that we did not make any explicit assumptions that the underlying consistency property of a recursive FC-normal NRS $\mathcal{S} = \langle U, N \rangle$ is any way effective. Indeed none of the above results require that the underlying consistency property has any effective properties. We define a consistency property Con for $\mathcal{S}$ to be **finitely decidable** if there is an effective procedure which applied to the canonical index of any finite subset X of U determines whether $X \in Con$. Then the following result easily follows from our completeness theorem for FC-normal NRS's.

Theorem 6.13 *Let $\mathcal{S} = \langle U, N \rangle$ be a highly recursive rule system such that $\mathcal{S}$ is FC-normal with respect to a decidable consistency property Con. Then $\{u \in U : u$ is in some extension of $\mathcal{S}\}$ is recursive.*

Finally we have the following result about recursive FC-normal nonmonotonic rule systems.

Theorem 6.14 *Let T be a recursive tree in $2^{<\omega}$ such that $[T] \neq \emptyset$. Then there is an FC-normal recursive NRS $\mathcal{S} = \langle U, N \rangle$ such that there is an effective one-to-one degree preserving correspondence between $[T]$ and $\mathcal{E}(\mathcal{S})$.*

Reiter ([Rei80]) proved that there is a recursive normal default theory with no recursive extension. Theorem 6.14 generalizes his result but also gives much finer information even for recursive normal default theories since the set of degrees of paths through highly recursive trees have been extensively studied. Our correspondence will allow us to transfer all results about the possible degrees of paths through highly recursive trees to results about the degrees of extensions of recursive FC-normal NRS systems. That is, all the results of Corollary 6.5 continue to hold if we replace recursive nonmonotonic rule system by FC-normal recursive nonmonotonic rule system in each part of its statement. Moreover, we note that the proof of Theorem 6.14 explicitly constructs recursive FC-normal NRS's such that the underlying consistency property is not finitely decidable.

Corollary 6.15 *1. There is an FC-normal recursive nonmonotonic rule system $\langle U, N\rangle$ such that $\langle U, N\rangle$ has $2^{\aleph_0}$ extensions but no recursive extensions.*

2. *There is an FC-normal recursive nonmonotonic rule system $\langle U, N\rangle$ such that $\langle U, N\rangle$ has $2^{\aleph_0}$ extensions and any two extensions $E_1 \neq E_2$ of $\langle U, N\rangle$ are Turing incomparable.*

3. *If $\mathbf{a}$ is any Turing degree such that $\mathbf{0} <_T \mathbf{a} \leq_T \mathbf{0}'$, then there is a highly recursive nonmonotonic rule system $\langle U, N\rangle$ such that $\langle U, N\rangle$ has $2^{\aleph_0}$ extensions but no recursive extensions and $\langle U, N\rangle$ has an extension of degree $\mathbf{a}$.*

4. *If $\mathbf{a}$ is any Turing degree such that $\mathbf{0} <_T \mathbf{a} \leq_T \mathbf{0}'$, then there is an FC-normal recursive nonmonotonic rule system $\langle U, N\rangle$ such that $\langle U, N\rangle$ has $\aleph_0$ extensions, $\langle U, N\rangle$ has an extension E of degree $\mathbf{a}$ and if $E' \neq E$ is an extension of $\langle U, N\rangle$, then E' is recursive.*

5. *There is an FC-normal recursive nonmonotonic rule system $\langle U, N\rangle$ such that $\langle U, N\rangle$ has $2^{\aleph_0}$ extensions and if $\mathbf{a}$ is the degree of any extension E of $\langle U, N\rangle$ and $\mathbf{b}$ is any recursively enumerable degree such that $\mathbf{a} <_T \mathbf{b}$, then $\mathbf{b} \equiv_T \mathbf{0}'$.*

6. *If $\mathbf{a}$ is any recursively enumerable Turing degree, then there is an FC-normal recursive nonmonotonic rule system $\langle U, N\rangle$ such that $\langle U, N\rangle$ has $2^{\aleph_0}$ extensions and the set of recursively enumerable degrees $\mathbf{b}$ which contain an extension of $\langle U, N\rangle$ is precisely the set of all recursively enumerable degrees $\mathbf{b} \geq_T \mathbf{a}$.*

Theorem 6.8 allows us to compare the FC-normal nonmonotonic rule systems (and thus FC-normal logic programs) with another class of rule systems that always have extensions.

To this end, motivated by Apt, Blair and Walker [ABW87] and Przymusinski [Prz88] consider the class of locally stratified rule systems. These are rule systems $\mathcal{S} = \langle U, N\rangle$ with the following property: There exists a function $\mathit{rank} : U \rightarrow \mathit{Ord}$ such that for every rule

$$\frac{\alpha_1, \ldots, \alpha_m : \beta_1, \ldots, \beta_n}{\gamma}$$

in N, for all $1 \leq i \leq m$, $\mathit{rank}(\alpha_i) \leq \mathit{rank}(\gamma)$ and for all $1 \leq j \leq n$, $\mathit{rank}(\beta_j) < \mathit{rank}(\gamma)$.

Apt, Blair and Walker [ABW87] (see also [MT93]) show that a stratified NRS possesses a unique extension. Hence, stratified NRS systems share with FC-normal NRS systems the property that an extension always exists. This similarity is, however, superficial. We will show that the classes of stratified and FC-normal systems are very different.

Example 6.1

1. There exists a locally stratified NRS system which is not FC-normal,
2. There exist FC-normal NRS system which is not stratified.

(1) It follows from the results of [BMS91] that for every recursive ordinal number α there exists a recursive locally stratified nonmonotonic rule system $\mathcal{S}_\alpha$ such that its unique extension has the degree $\geq_T 0^{(\alpha)}$. By Theorem 6.8 $\mathcal{S}_\alpha$ cannot be FC-normal for $\alpha > 3$, as every recursive FC-normal NRS system possesses an extension of the degree $\leq 0''$.
(2) Since there are FC-normal NRS systems with more than one extension, such systems cannot be locally stratified.

Next, we notice that the locally stratified have another property that makes their use for belief revision awkward. Namely, that the locally stratified NRS systems do not possess the semimonotonicity property.

Example 6.2 Let $U = \{a, \ldots, z\}$ and let N be composed of these three rules:

$$\frac{\,:r\,}{w} \qquad \frac{q:s}{r} \qquad \frac{p:\,}{q}$$

Then $\langle U, N\rangle$ is stratified and $\{w\}$ is its unique extension. Now add the rule

$$\frac{\,:t\,}{p}$$

The new system is again stratified, but its unique extension is now $\{p, q, r\}$. Thus semimonotonicity fails for stratified programs.

7 Proofs of general results on FC-normal Nonmonotonic Rule Systems

In this section, we shall give the proof of the results stated in Section 4.

¿From now on we assume all FC-normal nonmonotonic rule systems have the consistency property given by Con.

Theorem 4.1 *Every FC-normal nonmonotonic rule system has an extension.*

Proof: We shall show that our forward chaining construction will always produce an extension. Thus fix some well-ordering $\prec$ of $nmon(\mathcal{S})$. Our well-ordering $\prec$ determines some listing of the rules of $nmon(\mathcal{S})$,$\{r_\alpha : \alpha \in \gamma\}$, where γ is some ordinal. Let Θ_γ be the least cardinal such that $\gamma \leq \Theta_\gamma$. In what follows, we shall assume that the ordering among ordinals is given by $\in$. Recall our forward chaining construction an increasing sequence of sets $\{E^\prec_\alpha\}_{\alpha\in\Theta_\gamma}$ as follows.

The forward chaining construction of $E^\prec$.

Case 0. Let $E^\prec_0 = cl_{mon}(\emptyset)$.

Case 1. $\alpha = \eta + 1$ is a successor ordinal.
Given $E^\prec_\eta$, let $\ell(\alpha)$ be the least $\lambda \in \gamma$ such that $r_\lambda = \dfrac{\alpha_1, \ldots, \alpha_p : \beta_1, \ldots, \beta_k}{\psi}$ where $\alpha_1, \ldots, \alpha_p \in E^\prec_\eta$ and $\beta_1, \ldots, \beta_k$, $\psi \notin E^\prec_\eta$. If there is no such $\ell(\alpha)$, then let $E^\prec_{\eta+1} = E^\prec_\alpha = E^\prec_\eta$. Otherwise, let

$$E^\prec_{\eta+1} = E^\prec_\alpha = cl_{mon}(E^\prec_\eta \cup \{cln(r_{\ell(\alpha)})\}).$$

Case 2. α is a limit ordinal. Then let $E^\prec_\alpha = \bigcup_{\beta\in\alpha} E^\prec_\beta$.

We then let $E^\prec = \bigcup_{\alpha\in\Theta_\gamma} E^\prec_\alpha$.

It is straightforward to prove by (transfinite) induction that $Con(E^\prec_\alpha)$ holds for all $\alpha \in \Theta_\gamma$ and hence $Con(E^\prec)$ holds. Next we want to prove by (transfinite) induction that $E^\prec_\alpha \subseteq C_{E^\prec}(\emptyset)$ for all $\alpha \in \Theta_\gamma$. If $\alpha = 0$, then clearly $E^\prec_0 = cl_{mon}(\emptyset) \subseteq C_{E^\prec}(\emptyset)$. Suppose α is a successor ordinal and $\eta + 1 = \alpha$. Assume by induction that $E^\prec_\eta \subseteq C_{E^\prec}(\emptyset)$. Then if $E^\prec_\eta \neq E^\prec_{\eta+1}$, there exists one rule $r_{\ell(\eta+1)} = \dfrac{\alpha_1, \ldots, \alpha_p : \beta_1, \ldots, \beta_k}{\psi}$ where $\alpha_1, \ldots, \alpha_p \in E^\prec_\eta$, $\beta_1, \ldots, \beta_k \notin E^\prec_\eta$, and $E^\prec_{\eta+1} = cl_{mon}(E_\eta \cup \{\psi\})$. But since $r_{\ell(\eta+1)}$ is FC-normal, we know that $E^\prec_\eta \cup \{\psi, \beta_i\}$ is not consistent for all $i \leq k$. Since $E^\prec$ is consistent, it must be the case that $E^\prec_\eta \cup \{\psi, \beta_i\} \not\subseteq E^\prec$ for all $i \leq k$ since subsets of consistent sets are consistent. Thus for all $i \leq k$, $\beta_i \notin E^\prec$. Hence $r_{\ell(\eta+1)}$ shows that $\psi \in C_{E^\prec}(\emptyset)$. But then $E^\prec_\eta \cup \{\psi\} \subseteq C_{E^\prec}(\emptyset)$ from which it easily follows that $cl_{mon}(E^\prec_\eta \cup \{\psi\}) = E^\prec_{\eta+1} \subseteq C_{E^\prec}(\emptyset)$. For α a limit, we can assume by induction that $E^\prec_\beta \subseteq C_{E^\prec}(\emptyset)$ for all $\beta \in \alpha$ and so $E^\prec_\alpha = \bigcup_{\beta\in\alpha} E^\prec_\beta \subseteq C_{E^\prec}(\emptyset)$. Thus we have shown $E^\prec_\beta \subseteq C_{E^\prec}(\emptyset)$ for all $\beta \in \Theta_\gamma$ and hence $E^\prec = \bigcup_{\beta\in\Theta_\gamma} E^\prec_\beta \subseteq C_{E^\prec}(\emptyset)$.

To prove that $C_{E^\prec}(\emptyset) \subseteq E^\prec$, we proceed by induction on the length of minimal proof schemes. That is, suppose that if p is a minimal proof scheme of length $\leq m$

such that $\mathrm{supp}\,(p) \cap E^{\prec} = \emptyset$, then $cln(p) \in E^{\prec}$. Now suppose $q =<< \alpha_0, r_{\eta_0}, G_0 >, \ldots, < \alpha_m, r_{\eta_m}, G_m >>$ is a minimal proof scheme of length $m+1$ where $G_m \cap E^{\prec} = \emptyset$. Then by induction $\alpha_1, \ldots, \alpha_{m-1} \in E^{\prec}$ and hence $\alpha_1, \ldots, \alpha_{m-1} \in E^{\prec}_{\alpha}$ for some $\alpha \in \Theta_\gamma$. Suppose $r_{\eta_m} = \dfrac{\alpha_{i_0}, \ldots \alpha_{i_s} : \beta_1, \ldots, \beta_k}{\alpha_\eta}$ where $i_0 < \ldots < i_s < m$ and $\beta_1, \ldots, \beta_k \notin E^{\prec}$. Now it is easy to see that our construction ensures that if $r_{\ell(\eta+1)}$ is defined, then $cln(r_{\ell(\eta+1)}) \notin E^{\prec}_{\eta}$. Hence if $\lambda \neq \eta$ and $r_{\ell(\lambda)}$ and $r_{\ell(\eta)}$ are defined, then $r_{\ell(\lambda)} \neq r_{\ell(\eta)}$. Thus the function $\ell()$ is one-to-one on its domain. Now suppose that $\alpha_\eta \notin E^{\prec}$. Then for all $\lambda + 1$ greater than α, r_{η_m} is a candidate to be $r_{\ell(\lambda+1)}$ at stage $\lambda + 1$. Hence it must be that case that $r_{\ell(\lambda+1)}$ is defined and $\ell(\lambda + 1) \in \eta_m$. But this is impossible. That is, if Θ_γ is infinite, then the cardinality of $\{r_{\ell(\lambda+1)} : \eta_m \in \lambda \in \Theta_\gamma\}$ is equal to the cardinality of Θ_γ which is strictly greater than the cardinality of $\{\delta : \delta \in \eta_m\}$. Similary if Θ_γ is finite, then the fact that $r_{\ell(\lambda)}$ is defined for all $\lambda \leq \Theta_\gamma$ and $\ell()$ is one-to-one would mean that the cardinality of $\{r_{\ell(\lambda)} : \lambda \in \Theta_\gamma\}$ is equal to the cardinality of $nmon(\mathcal{S})$ so that every rule $r \in nmon(\mathcal{S})$ must be equal to $r_{\ell(\lambda)}$ for some λ. Thus in either case we have shown that if $\alpha_n \notin E^{\prec}$, then for some $\mu \in \Theta_\gamma$, r_{η_n} is the least rule $r = \dfrac{\delta_1, \ldots, \delta_s : \gamma_1, \ldots, \gamma_t}{\psi}$ such that $\delta_1, \ldots, \delta_k \in E^{\prec}_{\mu}$ and $\gamma_1, \ldots, \gamma_t$, $\psi \notin E^{\prec}_{\mu}$. But then by construction $\alpha_n \in E^{\prec}_{\mu+1} \subseteq E^{\prec}$. Thus α_n must be in $E^{\prec}$. Hence $C_{E^{\prec}}(\emptyset) \subseteq E^{\prec}$ and $E^{\prec}$ is an extension as claimed. □

Note that the proof of Theorem 4.1, remains unchanged if instead of starting with $cl_{mon}(\emptyset)$ at stage 0, we start with $cl_{mon}(I)$ where $I \in Con$. Thus we also have the following.

Theorem 4.2 *Let $\mathcal{S} = \langle U, N \rangle$ be an FC-normal nonmonotonic rule system with respect to consistency property Con. Let I be a subset of U such that $I \in Con$. Then there exists an extension I' of $\mathcal{S}$ such that $I \subseteq I'$.*

Next we want to show that every extension of an FC-normal NRS $\mathcal{S} = \langle U, N \rangle$ can be constructed by our forward chaining construction relative to an appropriate ordering of the $nmon(\mathcal{S})$.

Theorem 4.3 *If $\mathcal{S} = \langle U, N \rangle$ is an FC-normal NRS, and $\prec$ is* any *well-ordering of $nmon(\mathcal{S})$, then :*
(1) $E^{\prec}$ is an extension of $\mathcal{S}$.
(2) (completeness of the construction). Every extension of $\mathcal{S}$ is of the form $E^{\prec}$ for a suitably chosen ordering $\prec$ of $nmon(\mathcal{S})$.

Proof: (1) follows from our proof of Theorem 4.1.
(2) We prove the following fact:
Let F be an extension of an FC-normal NRS $\mathcal{S} = \langle U, N \rangle$. Let $\mu = \mathrm{card}\,(NG(F, \mathcal{S}))$ and let $\prec$ be some well-ordering of $nmon(\mathcal{S})$ such that the listing of $nmon(\mathcal{S})$ deter-

mined by $\prec$, $\{r_\alpha : \alpha \in \gamma\}$, is such that $\mu \leq \gamma$ and $NG(F, \mathcal{S}) = \{r_\alpha : \alpha \in \mu\}$. Then
(i) $F = cl_{mon}(\{cln(r) : r \in NG(F, \mathcal{S})\})$ and
(ii) $F = E^{\prec}$ where $E^{\prec}$ is constructed by our forward chaining construction.
For (i), note that for each $r = \dfrac{\alpha_1, \ldots, \alpha_n : \beta_1, \ldots, \beta_k}{\psi} \in NG(F, \mathcal{S})$, $cln(r) \in F$. Moreover for any set $W \subseteq C_F(\emptyset)$, $cl_{mon}(W) \subseteq C_F(\emptyset)$ so that $cl_{mon}(\{cln(r) : r \in NG(F, \mathcal{S})\}) \subseteq C_F(\emptyset)$. Then a straightforward induction on the length of a minimal proof scheme p will show that if $\mathrm{supp}\,(p) \cap F = \emptyset$, then $cln(p) \in cl_{mon}(\{cln(r) : r \in NG(F, \mathcal{S})\})$. It then follows that $C_F(\emptyset) = cl_{mon}(\{cln(r) : r \in NG(F, \mathcal{S})\})$.

For (ii), let $\{E^{\prec}_\alpha : \alpha \in \Theta_\gamma\}$ be constructed by the forward chaining construction relative to the well-ordering of rules $\{r_\alpha : \alpha \in \gamma\}$. Then we claim $E^{\prec}_\mu = F$ and $E^{\prec}_\alpha = E^{\prec}_\mu$ for $\alpha > \mu$. First it is easy to show by induction that $cl_{mon}(E^{\prec}_\alpha) = E^{\prec}_\alpha$ for all α. Next we claim that if $\alpha \in \mu$, then $E^{\prec}_\alpha \subseteq F$ and moreover if $E^{\prec}_\alpha \neq E^{\prec}_{\alpha+1}$, then $\ell(\alpha+1) \in \mu$. That is, $E^{\prec}_0 = cl_{mon}(\emptyset) \subseteq F$. Next suppose by induction that $E^{\prec}_\beta \subseteq F$ for all $\beta \in \alpha$. Then if α is a limit ordinal, $E^{\prec}_\alpha = \bigcup_{\beta \in \alpha} E^{\prec}_\beta \subseteq F$. If α is a successor ordinal, we can assume by induction that $E^{\prec}_\eta \subseteq F$ where $\eta + 1 = \alpha$. Now consider $E^{\prec}_\eta$. If $E^{\prec}_\eta = F$, then for any rule $r = \dfrac{\alpha_1, \ldots, \alpha_n : \beta_1, \ldots, \beta_k}{\psi}$ in $nmon(\mathcal{S})$, it must be the case that either $\{\beta_1, \ldots, \beta_k\} \cap F \neq \emptyset$, $\{\alpha_1, \ldots, \alpha_n\} \not\subseteq F$, or $\psi \in F$ since F is an extension. That is, if $E^{\prec}_\eta = F$, then $\ell(\eta+1)$ must be undefined and hence $E^{\prec}_\eta = E^{\prec}_{\eta+1} = F$. If $E^{\prec}_\eta \neq F$, then consider some $a \in F - E^{\prec}_\eta$. Since $a \in F$, there is some minimal proof scheme $p =<< \alpha_0, \overline{r}_0, G_0 >, \ldots, < \alpha_m, \overline{r}_m, G_m >>$ where $\alpha_m = a$ and $G_m \cap F = \emptyset$ which witnesses that $a \in F$. Since $a \notin E^{\prec}_\eta$, there must be some $k < m$ such that $\alpha_1, \ldots, \alpha_{k-1} \in E^{\prec}_\eta$ and $\alpha_k \notin E^{\prec}_\eta$. Then consider $\overline{r}_k = \dfrac{\alpha_{i_0}, \ldots, \alpha_{i_j} : \beta_1, \ldots, \beta_t}{\alpha_k}$ where $i_0 < \cdots < i_j < k$. Now it cannot be that $\{\beta_1, \ldots, \beta_t\} = \emptyset$ since otherwise $\alpha_k \in cl_{mon}(E^{\prec}_\eta) = E^{\prec}_\eta$. Thus $\{\beta_1, \ldots, \beta_t\} \neq \emptyset$. But since $\{\beta_1, \ldots, \beta_t\} \subseteq G_m$ and $G_m \cap F = \emptyset$, it must be the case that $\{\beta_1, \ldots, \beta_t\} \cap E^{\prec}_\eta = \emptyset$ and $\{\beta_1, \ldots, \beta_t\} \cap F = \emptyset$. Hence $\overline{r}_m \in NG(F, \mathcal{S})$ and $\overline{r}_m$ is a candidate to be $r_{\ell(\eta+1)}$. But this means that if $\overline{r}_m = r_\beta$ in our ordering of rules in $nmon(\mathcal{S})$, then $\ell(\eta+1) \leq \beta < \mu$. But for any $\delta \in \mu$, $cln(r_\delta) \in F$ by our choice of our well-ordering. Thus $cln(r_{\ell(\eta+1)}) \in F$ so that $E^{\prec}_\eta \cup \{cln(r_{\ell(\eta+1)})\} \subseteq F$ and hence $cl_{mon}(E^{\prec}_\eta \cup \{cln(r_{\ell(\eta+1)})\}) = E^{\prec}_{\eta+1} = E^{\prec}_\alpha \subseteq F$.

It follows that $E^{\prec}_\mu \subseteq F$ since $E_\mu = \bigcup_{\alpha \in \mu} E^{\prec}_\alpha$ and $E^{\prec}_\alpha \subseteq F$ for all $\alpha \in \mu$. We claim that it must be the case that $E^{\prec}_\mu = F$ for otherwise $E^{\prec}_\mu \subset F$ and hence for all $\alpha \in \mu$, $E^{\prec}_\alpha \subset F$. But our argument above shows that if $E^{\prec}_\alpha \subset F$, then $E^{\prec}_\alpha \subset E^{\prec}_{\alpha+1}$ and $\ell(\alpha+1) \in \mu$. This fact, in turn, will allow us to prove by induction on the length of a minimal proof scheme that for all $r \in NG(F, \mathcal{S})$, $cln(r) \in E^{\prec}_\mu$. That is, suppose $\psi = cln(r)$ for some $r \in NG(F, \mathcal{S})$. Now $\psi = cln(p)$ for some minimal proof scheme $p =<< \alpha_0, r_0, G_0 >, \ldots, < \alpha_m, r_m, G_m >>$ where $G_m \cap F = \emptyset$ and $\alpha_m = \psi$. Now assume by induction that for all φ such that $\varphi = cln(r)$ for some $r \in NG(F, \mathcal{S})$ and φ is the conclusion of some minimal proof scheme q such that $\mathrm{supp}\,(q) \cap F = \emptyset$ and length of $q < m$ is in $E^{\prec}_{\eta}$. Note that each r_k for $k < m$ is either in $mon(\mathcal{S})$ or in $NG(F, \mathcal{S})$ since p shows $\alpha_1, \ldots, \alpha_n \in F$ and $cons(r_k) \subseteq G_m$ where $G_m \cap F = \emptyset$. It follows that each α_i for $i < m$ such that $r_i \in NG(F, \mathcal{S})$ is in $E^{\prec}_{\mu}$ by our induction hypothesis. But then $\{\alpha_1, \ldots, \alpha_{m-1}\} \subseteq cl_{mon}(E^{\prec}_{\mu}) = E^{\prec}_{\mu}$. So consider r_m. Now if $r_m \in mon(\mathcal{S})$, then $\psi = \alpha_m \in cl_{mon}\{\alpha_1, \ldots, \alpha_{m-1}\} \subseteq E^{\prec}_{\mu}$. If $r_m \in NG(F, \mathcal{S})$, then $r_m = r_\xi$ in our orderings of rules where $\xi < \mu$. Moreover there is some $\lambda \in \mu$ such that $\{\alpha_1, \ldots, \alpha_{m-1}\} \subseteq E^{\prec}_{\lambda}$. But then for any $\lambda \leq \delta \leq \mu$, if $\psi \notin E^{\prec}_{\delta}$ then r_ξ is a possible candidate to be $r_{\ell(\delta+1)}$. Hence it must be that case that $r_{\ell(\delta+1)}$ is defined and $\ell(\delta+1) \in \xi$. But this is impossible. That is, if μ is infinite, then the cardinality of $\{r_{\ell(\delta+1)} : \lambda \in \delta \in \mu\}$ is equal to the cardinality of μ which is strictly greater than the cardinality of $\{\alpha : \alpha \in \xi\}$. Similary if μ is finite, then the fact that $r_{\ell(\lambda)}$ is defined for all $\lambda \leq \mu$ and $\ell()$ is one-to-one would mean that the cardinality of $\{r_{\ell(\lambda)} : \lambda \leq \mu\}$ is equal to the cardinality of μ so that every rule r_δ with $\delta \leq \mu$ must be equal to $r_{\ell(\lambda)}$ for some $\lambda \leq \mu$. Thus in either case we have shown that if $\psi \notin E^{\prec}_{\mu}$, then for some $\delta \leq \mu$, $r_\xi = r_{\ell(\delta)}$. But then by construction $\psi \in E^{\prec}_{\delta} \subseteq E^{\prec}_{\mu}$. Thus ψ must be in $E^{\prec}_{\mu}$. Thus we have shown that $\{cln(r) : r \in NG(F, \mathcal{S})\} \subseteq E^{\prec}_{\mu}$ if $E^{\prec}_{\mu} \subset F$. But this is a contradiction since by (i), $F = cl_{mon}(\{cln(r) : r \in NG(F, \mathcal{S})\}) \subseteq cl_{mon}(E^{\prec}_{\mu}) = E^{\prec}_{\mu}$. Thus it must be the case that $E^{\prec}_{\mu} = F$.

Note we have already shown that if $E^{\prec}_{\alpha} = F$, then $E^{\prec}_{\alpha} = E^{\prec}_{\alpha+1}$. Thus since $E^{\prec}_{\mu} = F$ it easily follows that $E^{\prec}_{\lambda} = F$ for all $\mu \leq \lambda \leq \Theta_\gamma$. Hence $E^{\prec} = \bigcup_{\alpha \in \Theta_\gamma} E_\alpha = F$ as claimed. □

Since every $E^{\prec}$ is in Con, we immediately get the following corollary.

Corollary 4.4 *Let $\mathcal{S} = \langle U, N \rangle$ be an FC-normal nonmonotonic rule system with respect to consistency property Con, then every extension of $\mathcal{S}$ is in Con.*

Next we show that if our FC-normal nonmonotonic rule system $\langle U, N \rangle$ is countable, i.e. if U is countable which automatically implies that N is countable, then every extension of $\langle U, N \rangle$ can be constructed via the countable forward chaining construction relative to some well-ordering $\prec$ of $nmon(\langle U, N \rangle)$ of the order type of ω.

Theorem 4.5 *If $\mathcal{S} = \langle U, N \rangle$ is a countable FC-normal nonmonotonic rule system, then*

1. *$E^{\prec}$ constructed via the countable forward chaining construction with respect to $\prec$, where $\prec$ is any* any *well-ordering of $nmon(\mathcal{S})$ of order type ω, is an extension of $\mathcal{S}$.*

2. *(completeness of the construction). Every extension of $\mathcal{S}$ is of the form $E^{\prec}$ for a suitably chosen ordering $\prec$ of $nmon(\mathcal{S})$ of order type ω where $E^{\prec}$ is constructed via the countable forward chaining construction.*

Proof. We note that if $\prec$ is a well-ordering of $nmon(\mathcal{S})$ of order type ω, the countable forward chaining algorithm is just the first ω steps of the forward chaining algorithm. Thus to prove (1), we must show that if we construct $E^{\prec}$ with respect the forward chaining algorithm, then $E^{\prec}_{\omega} = E^{\prec}_{\lambda}$ for all $\lambda \geq \omega$. In fact, we need only show that $E^{\prec}_{\omega} = E^{\prec}_{\omega+1}$. Now suppose that $r_{\ell(\omega+1)} = \frac{\alpha_1, \ldots, \alpha_n : \beta_1, \ldots, \beta_k}{\psi}$ is defined. Thus $\alpha_1, \ldots, \alpha_n \in E^{\prec}_{\omega}$ and $\beta_1, \ldots, \beta_k, \psi \notin E^{\prec}_{\omega}$. Moreover $r_{\ell(\omega+1)} = r_q$ for some q where $\{r_n\}_{n\in\omega}$ is the ordering of rules determined by $\prec$. But since $E^{\prec}_{\omega} = \bigcup_{n\in\omega} E^{\prec}_n$, there must be some s such that $\alpha_1, \ldots, \alpha_n \in E^{\prec}_s$. Hence for all $t \geq s$, $\beta_1, \ldots, \beta_k, \psi \notin E^{\prec}_t$ so that r_q is candidate to be $r_{\ell(t)}$ for all $t > s$. Since the function $\ell(t)$ is one-to-one, it easily follows that there would have to be some finite t such that $r_q = r_{\ell(t)}$. Thus $r_{\ell(\omega+1)}$ must not be defined and hence $E^{\prec}_{\omega} = E^{\prec}$.

Next we consider the proof of (2). Note that if we apply the proof of Theorem 4.3 to F in this case the most natural thing to do is to order the rules of $NG(F, \mathcal{S})$ first, say $NG(F, \mathcal{S}) = \{s_0, s_1, \ldots\}$, and then follow this ordering by a listing all the rules of $nmon(\mathcal{S}) - NG(F, \mathcal{S}) = \{t_0, t_1, \ldots\}$. Now if $NG(F, \mathcal{S})$ is finite, then our listing of rules determines a well-ordering $\prec$ of order type ω in which the proof of Theorem 4.3 shows that $F = E^{\prec}$. If $NG(F, \mathcal{S})$ is infinite, then our listing of rules determines a well-ordering $\prec$ of order type $\omega + \omega$. It then follows from the proof of Theorem 4.3 that

$$E^{\prec}_0 \subseteq E^{\prec}_1 \subseteq \ldots \subseteq E^{\prec}_{\omega} = E^{\prec}_{1+\omega} = \ldots .$$

and that $E^{\prec}_{\omega} = F$. The key point to note is that for any $r = \frac{\alpha_1, \ldots, \alpha_n : \beta_1, \ldots, \beta_k}{\psi}$ which is not in $NG(F, \mathcal{S})$, it must be the case that $\{\beta_1, \ldots, \beta_k\} \cap F \neq \emptyset$ or $\{\alpha_1, \ldots, \alpha_n\} \not\subseteq F$. But since $F = \bigcup_{i\in\omega} E^{\prec}_i$, it also follows that either
(a) for some i, $\{\beta_1, \ldots, \beta_k\} \cap E^{\prec}_i \neq \emptyset$ or
(b) for all j, $\{\alpha_1, \ldots, \alpha_n\} \not\subseteq E^{\prec}_j$.
In case (a) if we insert r between s_k and s_{k+1} where $k = \ell(i)$, then this change will have no effect on the construction of the $E^{\prec}_i$'s. That is, the construction of $E^{\prec}$ up to stage i can depends only on $s_0, \ldots, s_{\ell(i)}$ and hence we will get the same sets, $E^{\prec}_j$ for $j \leq i$, for any ordering which starts out $s_0, \ldots, s_{\ell(i)}$. Thus if we take the ordering $s_0, \ldots, s_{\ell(i)}, r, s_{1+\ell(i)}, \ldots$, then because $\{\beta_1, \ldots, \beta_k\} \cap E^{\prec}_i \neq \emptyset$, r is not a candidate to be to be $r_{\ell(k)}$ for any $k > i$ and hence the insertion of r does not effect the rest of the construction of $E^{\prec}$. In case (b), we can insert r anywhere in the initial ω part of the list and it will have no effect on the construction of the $E^{\prec}_i$'s for $i \in \omega$ because the premises of r are never contained in such $E^{\prec}_i$'s. In this way, we can see that it is possible to interleave all the r's in $nmon(\mathcal{S}) - NG(F, \mathcal{S})$ into the basic ordering $s_0, s_1, \ldots$ so as to create an ordering of order type ω but with out changing the sequence $E^{\prec}_0, E^{\prec}_1, \ldots$. Thus it will still be the case that $F = E^{\prec}_{\omega} = \bigcup_{i<\omega} E^{\prec}_i$. □

Theorem 4.6 follows immediately from the following result:

Theorem [Semi–monotonicity] *Suppose $\mathcal{S} = \langle U, N\rangle$ is an FC-normal NRS. Let $D \subseteq nmon(\mathcal{S})$. Then*

1. $\mathcal{S}' = (U, mon(\mathcal{S}) \cup D)$ *is FC-normal NRS and*
2. *if E' is an extension of $\mathcal{S}'$, then there is an extension E of $\mathcal{S}$ such that*
   (a) $E' \subseteq E$ *and*
   (b) $NG(E', \mathcal{S}') \subseteq NG(E, \mathcal{S})$

Proof: The fact that $\mathcal{S}'$ is an FC-normal NRS is an immediate consequence of our definitions. For part (ii), let $\mu =$ cardinality of $NG(E', \mathcal{S}')$ and choose a well-ordering of $NG(E', \mathcal{S}')$, $\{r_\alpha : \alpha \in \mu\}$. Then extend this well-ordering to a well-ordering $\{r_\alpha : \alpha \in \gamma\}$ of nmon(S). It follows that if $E^{\prec}$ is constructed via our forward chaining algorithm with respect to the well ordering $\prec$ determined by $\{r_\alpha : \alpha \in \gamma\}$, then proof of Theorem 4.3 shows $E' = E^{\prec}_\mu$ so that $E' \subseteq E^{\prec}$.

It remains to prove that $NG(E', \mathcal{S}') \subseteq NG(E^{\prec}, \mathcal{S})$. Now suppose

$$r = \frac{\alpha_1, \ldots, \alpha_n : \beta_1, \ldots, \beta_k}{\psi} \in NG(E', \mathcal{S}').$$

Then $\{\alpha_1, \ldots, \alpha_n\} \subseteq E' \subseteq E^{\prec}$ and $\{\beta_1, \ldots, \beta_k\} \cap E' = \emptyset$. But note that $Con(E')$ holds since $E' = E^{\prec}_\mu$. By Theorem 4.3 $E' = cl_{mon}(\{cln(r) : r \in NG(E', \mathcal{S}')\})$. Thus $\psi \in E'$ and hence by the FC-normality of r, $E' \cup \{\psi, \beta_i\}$ is not consistent for any $i = 1, \ldots, k$. But since $E^{\prec}$ is consistent, $E' \cup \{\psi, \beta_i\} \not\subseteq E^{\prec}$ for any $i = 1, \ldots, k$. Hence $\beta_i \notin E^{\prec}$ for all $i = 1, \ldots, k$ and $r \in NG(E^{\prec}, \mathcal{S})$. □

We prove now the result on the orthogonality of extensions.

Theorem 4.7 [Orthogonality of Extensions] *Let $\mathcal{S} = \langle U, N\rangle$ be an FC-normal NRS with respect to a consistency property Con. Then if E_1 and E_2 are two distinct extensions of $\mathcal{S}$, $E_1 \cup E_2 \notin Con$.*

Proof: By Theorem 4.3, $E = \bigcup_{\alpha \in \Theta_\gamma} E^{\prec}_\alpha$ where $\{E^{\prec}_\alpha\}_{\alpha \in \Theta_\gamma}$ is the sequence constructed by the forward chaining construction relative to some well ordering $\prec$ of nmon(S). Let α be the least ordinal such that $E^{\prec}_\alpha \subset F$ but $E^{\prec}_{\alpha+1} \not\subset F$. Note there must be such an α since otherwise $E \subseteq F$ and then by the minimality of extensions, $E = F$. Thus the rule $r_{\ell(\alpha+1)} = \dfrac{\alpha_1, \ldots, \alpha_n : \beta_1, \ldots, \beta_k}{\psi}$ is such that $\{\alpha_1, \ldots, \alpha_n\} \subseteq E^{\prec}_\alpha$, $\emptyset \neq \{\beta_1, \ldots, \beta_k\}$, $\{\beta_1, \ldots, \beta_k\} \cap E^{\prec}_\alpha = \emptyset$ and $E^{\prec}_{\alpha+1} = cl_{mon}(E^{\prec}_\alpha \cup \{\psi\})$. Since $E^{\prec}_{\alpha+1} \not\subseteq F$, it must be that $\psi \notin F$. But this means that $\beta_i \in F$ for some i since otherwise $r_{\ell(\alpha+1)} \in NG(F, \mathcal{S})$ which would imply that $\psi \in F$ because F is an extension. By the

FC-normality of $r_{\ell(\alpha+1)}$, $E^{\prec}_{\alpha} \cup \{\psi, \beta_i\}$ is not consistent. But since $E^{\prec}_{\alpha} \cup \{\psi, \beta_i\} \subseteq E \cup F$, $E \cup F$ is also not consistent. $\Box$

Theorem 4.8 *Suppose $\mathcal{S} = \langle U, N \rangle$ is an FC-normal NRS with respect to consistency property Con such that $cl_{mon}(\{cln(r) : r \in nmon(\mathcal{S})\})$ is in Con. Then $\mathcal{S}$ has a unique extension.*

Proof: For a contradiction, assume $\mathcal{S}$ has two distinct extensions, E_1 and E_2. Then by our proof of Theorem 4.3 , $E_i = cl_{mon}(\{cln(r) : r \in NG(E_i, \mathcal{S})\})$ for $i = 1, 2$. But then for $i = 1, 2$, $E_i \subseteq cl_{mon}\{cln(r) : r \in nmon(\mathcal{S})\}$. Thus $E_1 \cup E_2$ is contained in a consistent set so that $E_1 \cup E_2$ is consistent, contradicting Theorem 4.7. $\Box$

Theorem 4.9 *Let $\mathcal{S} = \langle U, N \rangle$ be an FC-normal NRS with respect to a consistency property Con. Then $\varphi \in U$ is an element of some extension of $\mathcal{S}$ if and only if φ has a consistent proof scheme with respect to Con.*

Proof: Clearly if $\beta \in E$ where E is an extension, then $Con(E)$ by Corollary 4.4. Thus since $\beta \in C_E(\emptyset)$, there is a consistent minimal proof scheme for φ.

Conversely suppose $p =<< \varphi_0, r_0, G_0 >, \ldots, < \varphi_m, r_m, G_m >>$ is a consistent minimal proof scheme for β. Let $0 \leq i_1 < \cdots < i_k \leq m$ be set of all $i \leq m$ such that $r_i \in nmon(\mathcal{S})$. Now well-order nmon(S) so that $r_i, \ldots, r_{i_k}$ are the first k elements in the list. Then if we construct an extension via our forward chaining construction, it is easy to show by induction on k that $\beta \in E^{\prec}_k$. Hence $\beta \in E^{\prec}$ which is an extension. $\Box$

Theorem 4.10 *Suppose $\mathcal{S} = \langle U, N \rangle$ is an FC-normal NRS and that $D \subseteq nmon(\mathcal{S})$. Suppose further that E'_1 and E'_2 are distinct extensions of $(U, D \cup mon(\mathcal{S}))$. Then $\mathcal{S}$ has distinct extensions E_1 and E_2 such that $E'_1 \subseteq E_1$ and $E'_2 \subseteq E_2$.*

Proof: By Theorem 4.9, we know that there are extensions of $\mathcal{S}$, E_1 and E_2, such that $E'_1 \subseteq E_1$ and $E'_2 \subseteq E_2$. But then the orthogonality of extensions for $(U, D \cup mon(\mathcal{S}))$ ensures $E'_1 \cup E'_2$ is not consistent. Hence $E_1 \cup E_2$ is not consistent so that $E_1 \neq E_2$. $\Box$

8 Recursive FC-normal NRS Systems and the Complexity of their Extensions

In this section, we shall give all the proofs of Theorems stated in Section 6.

Our first result will show that under the assumption of FC-normality that once again even recursive NRS systems are guaranteed to have at least one relatively well behaved extension.

Theorem 6.8 *Suppose that $\mathcal{S} = \langle U, N \rangle$ is a recursive nonmonotonic rule system and $\mathcal{S}$ is FC-normal. Then $\mathcal{S}$ has an extension E such that E is r.e. in $0'$ and hence $E \leq_T 0''$.*

Proof: It is easy to see that since N is recursive, $mon(\mathcal{S})$ is also recursive. It then easily follows that if $X \subseteq U$ and X is r.e., then $cl_{mon}(X)$ is also r.e. In fact, there is a recursive function f such that for all e, $W_{f(e)} = cl_{mon}(W_e)$ where W_e is the e^{th} r.e. set $= \{x : \varphi_e(x) \downarrow\}$. We shall show that for any recursive well-ordering $\prec$ of $nmon(\mathcal{S})$ of order type ω, $E^{\prec}$ is r.e. in $0'$ where $E^{\prec}$ is constructed via the countable forward chaining algorithm with respect to $\prec$. That is, $\prec$ determines an effective listing of nmon(S), $r_0, r_1, \ldots$. Then the countable forward chaining construction relative to $\prec$ constructs a sequence of sets, $E_0^{\prec} \subseteq E_1^{\prec} \subseteq \ldots$. It is straightforward to prove by induction that $E_n^{\prec}$ is r.e. for all e. That is, suppose $E_n^{\prec} = W_{e_n}$ for some e_n. Then either $E_n^{\prec} = E_{n+1}^{\prec}$ or $E_n^{\prec} \neq E_{n+1}^{\prec}$ so that $E_{n+1}^{\prec} = cl_{mon}(E_n^{\prec} \cup \{cln(r_{\ell(n+1)}\}) = cl_{mon}(W_{e_n} \cup \{cln(r_{\ell(n+1)}\})$ which is clearly r.e.. Now there are two cases. Either there is an n such that $E_n^{\prec} = E_{n+1}^{\prec}$ in which case $E_n^{\prec} = E^{\prec}$ so that $E^{\prec}$ is r.e. Otherwise, $E_n^{\prec} \neq E_{n+1}^{\prec}$ for all n and hence $r_{\ell(n+1)}$ is defined for all n. In this case given an $0'$–oracle, we can test for membership in $E_n^{\prec}$ and hence it is easy to see that given $r_k = \dfrac{\alpha_1, \ldots, \alpha_s : \beta_1, \ldots, \beta_t}{\psi} \in nmon(\mathcal{S})$, we can test whether $\alpha_1, \ldots, \alpha_s \in E_n^{\prec}$, $\beta_1, \ldots, \beta_k$, $\psi \notin E_n^{\prec}$. Thus given an $0'$–oracle, we can effectively find $r_{\ell(n+1)}$. However given $r_{\ell(n+1)}$ and an r.e. index e_n such that $W_{e_n} = E_n^{\prec}$, then we can effectively produce an index r.e. index e_{n+1} such that $W_{e_{n+1}} = cl_{mon}(E_n^{\prec} \cup \{cln(r_{\ell(n+1)})\}) = E_{n+1}^{\prec}$. Thus there is an $0'$–effective sequence $e_0, e_1, \ldots$ such that $E_n^{\prec} = W_{e_n}$ for all n. Hence $E^{\prec} = \bigcup_n E_n^{\prec}$ is r.e. $0'$ in this case. Thus in either case, $E^{\prec}$ will be r.e. in $0'$. □

Note that Theorem 6.8 is in great contrast to the case of arbitrary recursive NRS $\mathcal{S} = \langle U, N \rangle$ where, for example as in Theorem 6.7 (a), there exists a recursive NRS $\mathcal{S}'$ such that $\mathcal{S}'$ has an extension but $\mathcal{S}'$ has no hyperarithmetic extension.

Theorem 6.10 *Let $\mathcal{S} = \langle U, N \rangle$ be a recursive rule system such that $\mathcal{S}$ is FC-normal and $nmon(\mathcal{S})$ is finite, then every extension of $\mathcal{S}$ is r.e.*

Proof: This Theorem follows immediately from our argument in Theorem 6.8 once we observe that if $nmon(\mathcal{S})$ is finite, then there is some finite stage such that $E_n^{\prec}$ equals $E^{\prec}$ for all well-ordering $\prec$ of $nmon(\mathcal{S})$. □

Another situation where a recursive FC-normal NRS $\mathcal{S} = \langle U, N \rangle$ is guaranteed to have an r.e. extension is when $\mathcal{S}$ is monotonically decidable. In particular if $mon(\mathcal{S})$ is finite, then $\mathcal{S} = \langle U, N \rangle$ is guaranteed to have an r.e. extension.

Theorem 6.11 *Let $\mathcal{S} = \langle U, N\rangle$ be a recursive rule system such that $\mathcal{S}$ is FC-normal and monotonically decidable, then $\mathcal{S}$ has an extension which is r.e..*

Proof: We shall show exactly as in Theorem 6.8 that for any recursive well-ordering $\prec$ of $nmon(\mathcal{S})$ of order type ω, $E^{\prec}$ is r.e. where $E^{\prec}$ is constructed via the countable forward chaining algorithm with respect to $\prec$. Again, there are two cases. If there is some n such that $E^{\prec}_n = E^{\prec}_{n+1}$, then $E^{\prec}_n = E^{\prec}$ and $E^{\prec}$ is r.e. Otherwise, $E^{\prec}_n \neq E^{\prec}_{n+1}$ for all n and hence $r_{\ell(n+1)}$ is defined for all n. In this case, it is easy to prove by induction that $E^{\prec}_n = cl_{mon}(\{r_{\ell(1)}, \ldots, r_{\ell(n)}\})$ for all $n \geq 1$. Because $\mathcal{S}$ is monotonically decidable, it follows that $E^{\prec}_n$ is recursive for all n. Indeed there is a recursive function f such that $\varphi_{f(k)}$ is characteristic function of $cl_{mon}(D_k)$. Thus given $\{r_{\ell(1)}, \ldots, r_{\ell(n)}\}$, we can effectively find k_n such that $D_{k_n} = \{r_{\ell(1)}, \ldots, r_{\ell(n)}\}$ and then $\varphi_{f(k_n)}$ will be the characteristic function of $E^{\prec}_n$. But then we can use $\varphi_{f(k_n)}$ to effectively find $r_{\ell(n+1)}$. Thus we can effectively find the sequence of rules $r_{\ell(1)}, r_{\ell(2)}, \ldots$ and since $E^{\prec} = cl_{mon}(\{r_{\ell(1)}, r_{\ell(2)}, \ldots\})$, $E^{\prec}$ is r.e.. $\square$

Next we consider the case where $\mathcal{S} = \langle U, N\rangle$ is a finite FC-normal nonmonotonic rule system, i.e we assume that both U and N are finite. In this case, we shall see that our forward chaining algorithm runs in polynomial time.

For complexity considerations, we shall assume that the elements of U are coded by strings over some finite alphabet Σ. Thus every $a \in U$ will have some length which we denote by $||a||$. Next, for a rule $r = \dfrac{a_1, \ldots, a_n : b_1, \ldots, b_m}{c}$, we define $||r|| = (\sum_{i \leq n} ||a_i||) + (\sum_{i \leq m} ||b_j||) + ||c||$. Finally, for a set Q of rules, we define

$$||Q|| = \sum_{r \in Q} ||r||.$$

Theorem 6.12 *Suppose $\mathcal{S} = \langle U, N\rangle$ is a finite FC-normal nonmonotonic rule system and $\prec$ is some well-ordering of $nmon(S)$. Then $E^{\prec}$ as constructed via our forward chaining algorithm can be computed in time $O(||mon(\mathcal{S})|| \cdot ||nmon(\mathcal{S})|| + ||nmon(\mathcal{S})||^2)$.*

Proof. First observe that our forward chaining algorithm will stop after stage n where $n = |nmon(\mathcal{S})|$ since $\ell(k)$ is a one-to-one function, i.e. at then end of stage n, there will be no possible candidate for $r_{\ell(n+1)}$ so that $E^{\prec}_n = E^{\prec}_{n+1} = E^{\prec}$.

Next consider a stage $k+1$ in the forward chaining construction. Given $E^{\prec}_k$, we must make a pass in order through the rules to check for each rule $r = \dfrac{a_1, \ldots, a_n, b_1, \ldots, b_m}{c}$ whether $\{a_1, \ldots, a_n\} \subseteq E^{\prec}_k$ and $\{b_1, \ldots, b_m, c\} \cap E^{\prec}_k = \emptyset$. Notice that at a cost of maintaining an appropriate data structure we can perform this check in $C||r||$ steps for some constant C and we call such a check a *rule check*. Now assuming that we process the rules in order, if r is the first rule such that $\{a_1, \ldots, a_n\} \subseteq E^{\prec}_k$

and $\{b_1, \ldots, b_m, c\} \cap E_k^{\prec} = \emptyset$, then $r = r_{k+1}, E_{k+1}^{\prec} = cl_{mon}(\{c\} \cup E_k^{\prec})$. Moreover if $\{b_1, \ldots, b_m, c\} \cap E_k^{\prec} \neq \emptyset$, then we know that r can never be a candidate to r_j for any $j > k$ so that we can just mark rule r and never consider it again. Of course we also mark r_{k+1} at stage $k+1$ if it is defined so that at each stage we will mark at least one $r \in nmon(\mathcal{S})$. Moreover, if r_{k+1} is not defined, then we can stop since then we know $E_k^{\prec} = E^{\prec}$.

It follows that at stage $k+1$, we need to look at most $|nmon(\mathcal{S})| - k$ rules and hence perform at most $|nmon(\mathcal{S})| - k$ rule checks. Since the construction must stop at stage $|nmon(\mathcal{S})|$, it follows that the entire construction requires at most

(a) $\binom{|nmon(\mathcal{S})| + 1}{2}$ rule checks,

(b) $|nmon(\mathcal{S})|$ operations of computing $cl_{mon}(E_k^{\prec} \cup \{c\})$ and

(c) the computation of $cl_{mon}(\emptyset)$.

Now if $||nmon(\mathcal{S})|| = k|nmon(\mathcal{S})|$ for some k, then the rule checks could require $O(||nmon(\mathcal{S})||^2)$ steps. Next consider the computations $cl_{mon}(A)$ which are required for **(b)** and **(c)** above. We claim all this can be done $O(||mon(\mathcal{S})|| \cdot ||nmon(\mathcal{S})||)$ steps. Since in our construction all the elements of $E_k^{\prec}$ must appear in one of the rules, we can assume $||A|| \leq ||mon(\mathcal{S})|| + ||nmon(\mathcal{S})||$. Now we can first make a pass through all the rules of $nmon(\mathcal{S})$ to get a list of all the elements of U which occur in one of the rules. Call this set V. Another pass through the rules will allow us to set up a system of pointers from each $c \in V$ to the set of rules $r \in mon(\mathcal{S})$ such that c occurs in the set of premises of r. We can also mark which c are in A. All this will require $C_1(||mon(\mathcal{S})|| + ||nmon(\mathcal{S})||) \leq C_1(||mon(\mathcal{S})|| \cdot ||nmon(\mathcal{S})||)$ steps. Now for each $c \in A$, use the pointers from c to the rules $r \in mon(\mathcal{S})$ to update each r by marking each premise of r in A. Now if a rule $r \in mon(\mathcal{S})$ has all of its premises marked, we mark the conclusion of r, *i.e.*, we add the $cln(r)$ to $cl_{mon}(A)$ and use the pointers from $cln(r)$ to rules in $mon(\mathcal{S})$ to further update the premises of each rule by marking $cln(r)$. We continue in this fashion until there are no more rules to update in which case A together with the marked conclusions will form the $cl_{mon}(A)$. Now assuming that updates can be performed in constant time, each rule $r \in mon(\mathcal{S})$ can require at most $||r||$ updates in this process since once all the premises of a rule have been marked we no longer have to consider it. Thus we require at most $||mon(\mathcal{S})||$ updates so the entire process takes at most $O(||mon(\mathcal{S})||)$ steps. Thus to compute the monotonic closures required in **(b)** and **(c)** above takes $O((1 + |nmon(\mathcal{S})|) \cdot ||mon(\mathcal{S})||) \leq O(||mon(\mathcal{S})|| \cdot ||nmon(\mathcal{S})||)$ steps. □

We pause to make one further observation about any highly recursive FC-normal NRS $\mathcal{S} = \langle U, N \rangle$ for which the underlying consistency property Con is finitely decidable. That is, if $\mathcal{S}$ is highly recursive, then we can effectively find the set of minimal proof schemes for any $x \in U$. If Con is finitely decidable, then we can effectively tell if any of the proof schemes for x are consistent. By Theorem 4.9, x has a consistent proof scheme if and only if x is in some extension of $\mathcal{S}$. Thus in this case, we can effectively decide whether x in some extension of $\mathcal{S}$. Thus we have proved the following result.

Theorem 6.13 *Let $\mathcal{S} = \langle U, N \rangle$ be a highly recursive rule system such that $\mathcal{S}$ is FC-normal with respect to a decidable consistency property Con. Then $\{u \in U : u$ is in some extension of $\mathcal{S}$ is recursive.*

We end this section with a result that shows that recursive FC-normal NRS's are at least as expressive as highly recursive NRS.

Theorem 6.14 *Let T be a recursive tree in $2^{<\omega}$ such that $[T] \neq \emptyset$. Then there is an FC-normal recursive NRS $\mathcal{S} = \langle U, N \rangle$ such that there is an effective one-to-one degree preserving correspondence between $[T]$ and $\mathcal{E}(\mathcal{S})$.*

Proof: We can assume T is $(0,2)$–tree, i.e., that for all $\alpha \in T$ either α is a terminal node or both $\alpha\hat{}0$ and $\alpha\hat{}1$ are in T. For if T is not a $(0,2)$–tree simply replace T by $T^* = T \cup \{\alpha\hat{}0, \alpha\hat{}1 : \alpha \in T\}$. It is then easy to see T^* is a recursive $(0,2)$–tree such that $[T] = [T^*]$. So assume T is a recursive $(0,2)$–tree $\subseteq 2^{<\omega}$ such that $[T] \neq \emptyset$.

We let $U = \{\emptyset\} \cup \{\alpha, \overline{\alpha} : \alpha \in T - \{\emptyset\}\}$. We let N consist of the following 9 classes of rules.

(1) $\frac{:}{\emptyset}$.

(2) $\frac{:}{\alpha}$ for all $\alpha \in T$ which are terminal nodes.

(3) $\frac{\overline{\alpha}:}{\alpha\hat{}0}$ $\quad$ $\frac{\overline{\alpha}:}{\alpha\hat{}1}$ for all $\alpha \in T$ such that α is not a terminal node.

(4) $\frac{\overline{\alpha\hat{}0}, \overline{\alpha\hat{}1}:}{\overline{\alpha}}$ for all $\alpha \in T$ such that α is not a terminal node.

(5) $\frac{\alpha:\alpha\hat{}0}{\alpha\hat{}1}$ $\quad$ $\frac{\alpha:\alpha\hat{}1}{\alpha\hat{}0}$ for all $\alpha \in T$ such that α is not a terminal node.

(6) $\frac{\alpha\hat{}1:}{\alpha\hat{}0}$ $\quad$ $\frac{\alpha\hat{}0}{\alpha\hat{}1}$ for all $\alpha \in T$ such that α is not a terminal node.

(7) $\frac{\alpha, \overline{\alpha}:}{\varphi}$ for all $\alpha \in T - \{\emptyset\}$ and all $\varphi \in U$.

(8) $\frac{\alpha:}{\beta}$ for all $\alpha \in T$ and $\beta \sqsubset \alpha$.

(9) $\frac{\alpha, \overline{\alpha\hat{}0}:}{\alpha\hat{}1}$ $\quad$ $\frac{\alpha, \overline{\alpha\hat{}1}:}{\alpha\hat{}0}$ for all $\alpha \in T$ such that α is not a terminal node.

It is easy to see that $\mathcal{S} = \langle U, N \rangle$ is a recursive nonmonotonic rule system. Given a path $\pi = (\pi(0), \pi(1), \ldots)$ through T, let $E_\pi = \{\alpha : \alpha$ is a node on $\pi\}$ $\cup \{\overline{\alpha} : \alpha$ is not a node on $\pi\}$. We claim that E is an extension of $\mathcal{S}$ if and only if $E = E_\pi$ for some $\pi \in [T]$. It is easy to see that E_π is an extension for every $\pi \in [T]$. That is, (1) shows that $\emptyset \in C_{E_\pi}(\emptyset)$ and repeated use of the rules in (5) will allow us to show that $(\pi(0), \ldots, \pi(n)) \in C_{E_\pi}(\emptyset)$ for all n. Then the rules in (6) will allow us to show that for every n, $\overline{(\pi(0),\ldots,\pi(n-1),\delta(n))} \in C_{E_\pi}(\emptyset)$ where

$$\delta(n) = \begin{cases} 0 & \text{if} \quad \pi(n) = 1 \\ 1 & \text{if} \quad \pi(n) = 0. \end{cases}$$

Now suppose $\alpha \in T$ and α is not on π. Then there is an n such that $(\pi(0),\ldots,\pi(n-1),\delta(n)) \sqsubseteq \alpha$. Hence repeated use of the rules in (3) will allow us to show $\overline{\alpha} \in C_{E_\pi}(\emptyset)$. Thus $E_\pi \subseteq C_{E_\pi}(\emptyset)$.

A straightforward proof by induction on the length of a derivation will show that $C_{E_\pi}(\emptyset) \subseteq E_\pi$. Hence $E_\pi = C_{E_\pi}(\emptyset)$ and E_π is an extension of $\mathcal{S}$.

Next assume that E is an extension. First we claim that $E \neq U$. For if $E = U$, then consider a node $\eta \in T - \{\emptyset\}$. If U is an extension, there must be a minimal proof scheme $p =<< \alpha_0, r_0, G_0 >, \ldots, < \alpha_m, r_m, G_m >>$ such that $\alpha_m = \eta$ and $G_m \cap U = \emptyset$, i.e., $G_m = \emptyset$. Pick η so that η has a minimal proof scheme q with the shortest possible length of all $\alpha \in C_U(\emptyset) - \{\emptyset\}$ and assume p is a minimal proof with the smallest possible length for η. But note the only rules which have conclusion η and has an empty set of constraints are the rules in (7) or (8). Thus $r_m = \frac{\alpha, \overline{\alpha}:}{\eta}$ for some $\alpha \in T$ or $r_m = \frac{\alpha:}{\eta}$ for some $\alpha \sqsupset \eta$. But this contradicts our choice of η since α must have a shorter minimal proof scheme than η. It follows that if $|\eta| \geq 1$, then $\eta \notin C_U(\emptyset)$ and hence U is not an extension. Once we know that U is not an extension, the rules in (7) show that for any $\alpha \in T - \{\emptyset\}$, E can contain at most one of α and $\overline{\alpha}$. Now assume by induction on n that there is a node $\beta_n = (\beta(0),\ldots,\beta(n-1))$ of length n such that β_n is on some infinite path through T and for all $\alpha \in T$ with $|\alpha| \leq n$, $\alpha \in E$ if and only if $\alpha \sqsubseteq \beta_n$ and $\overline{\alpha} \in E$ if and only if $\alpha \not\sqsubseteq \beta_n$. Now the rules in (3) ensure that if $\gamma \in T$ is such that $|\gamma| > n$ and γ does not extend β_n, then $\overline{\gamma} \in E$. Thus the only possible nodes, γ of length $n+1$ such that $\gamma \in E$ are $\gamma = \beta_n\hat{\ }0$ and $\gamma = \beta_n\hat{\ }1$. Because we are assuming that β_n is on an infinite path through T, we know that at least one of $\beta_n\hat{\ }0$ and $\beta_n\hat{\ }1$ must be on an infinite path through T. Now suppose $\beta_n\hat{\ }0$ is not on an infinite path through T. Then $A_0 = \{\eta : \eta \sqsupseteq \beta_n\hat{\ }0 \ \&\ \eta \in T\}$ is finite by König lemma. Note if $\eta \in A_0$ and η is a terminal node $\overline{\eta} \in E$ by (2). But then it is easy to see that since T is a $(0,2)$–tree, repeated use of the rules of (4) will allow us to show $\overline{\eta} \in E$ for all $\eta \in A_0$. In particular, $\overline{\beta_n\hat{\ }0} \in E$ and hence by rule (9), $\frac{\beta_n, \overline{\beta_n\hat{\ }0}:}{\beta_n\hat{\ }1}$, $\beta_n\hat{\ }1 \in E$. Thus we have shown that if $\beta_n\hat{\ }0$ is not on an infinite path through T, then $\beta_n\hat{\ }1 \in E$, $\beta_n\hat{\ }1$ is on an infinite path through T, and for all $\eta \in T$ such that $|\eta| \leq n+1$, $\eta \in E$ if and only if $\eta \sqsubseteq \beta_n\hat{\ }1$ and $\overline{\eta} \in E$ if and only if $\eta \sqsubseteq \beta_n\hat{\ }1$. Thus our inductive hypothesis holds at $n+1$ with $\beta_{n+1} = \beta_n\hat{\ }1$. A similar argument will show that our inductive hypothesis holds at $n+1$ with $\beta_{n+1} = \beta_n\hat{\ }0$ if $\beta_n\hat{\ }1$ is not on an infinite path through T. Thus we can assume that both $\beta_n\hat{\ }0$ and $\beta_n\hat{\ }1$ lie on infinite paths through T. Now consider the rules $R_0 = \frac{\beta_n : \beta_n\hat{\ }0}{\beta_n\hat{\ }1}$ and $R_1 = \frac{\beta_n : \beta_n\hat{\ }1}{\beta_n\hat{\ }0}$ from (5). Note that R_0 shows $\beta_n\hat{\ }1 \in E$ if $\beta_n\hat{\ }0 \notin E$ and R_1 shows $\beta_n\hat{\ }0 \in E$ if $\beta_n\hat{\ }1 \notin E$. Thus at least one of $\beta_n\hat{\ }0$ and $\beta_n\hat{\ }1$ must be in E. We claim that it cannot be the case that both $\beta_n\hat{\ }0$ and $\beta_n\hat{\ }1$ are in E. For if $\beta_n\hat{\ }0, \beta_n\hat{\ }1 \in E$, then consider the minimal proof scheme

$$p =<< \alpha_0, r_0, G_0 >, \ldots, < \alpha_m, r_m, G_m >>$$

of shortest possible length such that $G_m \cap E = \emptyset$ and $\alpha_m = \beta_n\hat{}0$. Note that any rule r with conclusion $\beta_n\hat{}0$ is either R_1 or comes from (7) or (8). Now $r_m \neq R_1$ since R_1 is blocked for E if $\{\beta_n\hat{}0, \beta_n\hat{}1\} \subseteq E$. Since $U \neq E$, we cannot use any rule from (7). Thus $r_m = \frac{\gamma:}{\beta_n\hat{}0}$ where $\gamma \sqsupset \beta_n\hat{}0$. It follows that $\gamma = \alpha_i$ for some i. It cannot be that $i < m-1$ since otherwise the proof scheme

$$p' = << \alpha_0, r_0, G_m >, \ldots, < \alpha_i, r_i, G_i >, < \beta_n\hat{}0, \frac{\gamma:}{\beta_n\hat{}0}, G_i >>$$

would violate our choice of p since it could be refined to a minimal proof scheme p'' of length $<$ the length of p such that $\mathrm{cl}\, n(p'') = \beta_n\hat{}0$ and $\mathrm{supp}\,(p'') \cap E = \emptyset$. Thus $\gamma = \alpha_{m-1}$. But then $r_{m-1} = \frac{\delta:}{\gamma}$ where $\gamma \sqsubset \delta$ from (8) or $r_{m-1} = \frac{\alpha:\alpha\hat{}(1-i)}{\alpha\hat{}i}$ where $\alpha\hat{}i = \gamma$ from (5). If $r_{m-1} = \frac{\delta:}{\gamma}$, then $\delta = \alpha_j$ for some $j < m-1$ and as before, the proof scheme $<< \alpha_0, r_0, G_0 >, \ldots, < \alpha_j, r_j, G_j >, < \beta_n\hat{}0, \frac{\delta:}{\beta_n\hat{}0}, G_j >>$ would violate our choice of p. If $r_{m-1} = \frac{\alpha:\alpha\hat{}(1-i)}{\alpha\hat{}i}$, then either (i) $\alpha \sqsupset \beta_n\hat{}0$ and $\alpha = \alpha_j$ for some $j < m-1$ or (ii) $\alpha = \beta_n\hat{}0$ and $\beta_n\hat{}0 = \alpha_j$ for some $j < m-1$. In either case, we would violate our choice of p. Thus there can be no such p and hence $\beta_n\hat{}0 \notin C_E(\emptyset)$ if $\{\beta_n\hat{}0, \beta_n\hat{}1\} \subseteq E$. Hence if $\{\beta_n\hat{}0, \beta_n\hat{}1\} \subseteq E$, $E \neq C_E(\emptyset)$ which contradicts the fact that E is an extension. Thus we must conclude that exactly one of $\beta_n\hat{}0$ and $\beta_n\hat{}1$ is in E. Now if $\beta_n\hat{}0 \in E$, then the rule $\frac{\beta_n\hat{}0:}{\overline{\beta_n\hat{}1}}$ from (9) shows $\overline{\beta_n\hat{}1} \in E$ and our induction hypothesis holds at $n+1$ with $\beta_{n+1} = \beta_n\hat{}0$. If $\beta_n\hat{}1 \in E$, then the rule $\frac{\beta_n\hat{}1:}{\overline{\beta_n\hat{}0}}$ shows that $\overline{\beta_n\hat{}0} \in E$ and our induction hypothesis holds at $n+1$ with $\beta_{n+1} = \beta_n\hat{}1$. This completes our induction and shows that $\emptyset = \beta_0 \sqsubset \beta_1 \sqsubset \beta_2 \sqsubset \ldots$ determines a path Π through T and that $E = E_\pi$.

It now follows that the corresponding $\pi \mapsto E_\pi$ is our desired effective one-to-one degree preserving correspondence between $\mathcal{P}(\mathcal{T})$ and $\mathcal{E}(\mathcal{S})$. Thus to complete our proof of the theorem, we need only check that $\mathcal{S}$ is FC-normal with respect to some consistency property. We define $X \subseteq U$ to be consistent, $Con(X)$, if and only if $X \subseteq E_\pi$ for some $\pi \in [T]$. It is easy to check that properties (1), (2), and (4) hold for Con. For property (3), note that if $X \subseteq E_\pi$, then $cl_{mon}(X) \subseteq C_{E_\pi}(\emptyset) = E_\pi$ since E_π is an extension. Thus Con defines a consistency property. Finally we must check that all rules $r \in nmon(\mathcal{S})$ are FC-normal with respect to Con. Note the only rules $r \in nmon(\mathcal{S})$ are of the form $\frac{\alpha:\alpha\hat{}0}{\alpha\hat{}1}$ and $\frac{\alpha:\alpha\hat{}1}{\alpha\hat{}0}$ from (5). Now clearly $\{\alpha\hat{}0, \alpha\hat{}1\}$ is not consistent so that all we need to show is that if $X \subset U$ is such that $Con(X)$, $cl_{mon}(X) = X$, $\alpha \in X$, $\alpha\hat{}(1-i) \notin X$, then $X \cup \{\alpha\hat{}i\}$ is consistent. Now consider the possibilities for a monotonically closed consistent set X. For some $\pi = (\pi(0), \pi(1), \ldots) \in [T]$, $X \subseteq E_\pi = \{\alpha : \alpha \text{ is on } \pi\} \cup \{\overline{\alpha} : \alpha \text{ is not on } \pi\}$. First suppose for infinitely many n, $(\pi(0), \ldots, \pi(n)) \in X$. Then rules of the form (8) show $(\pi(0), \ldots, \pi(m)) \in X = cl_{mon}(X)$ for every m. Next rules of the form (6) allow us to show $\overline{(\pi(0),\ldots,\pi(n-1),1-\pi(n))} \in X = cl_{mon}(X)$ for all n and then rules of the form (3) will allow us to show $\overline{\alpha} \in X = cl_{mon}(X)$ for all α not on π. That is, $X = E_\pi$ if for infinitely many n, $(\pi(0),\ldots,\pi(n)) \in X$. But in such a case, $\alpha \in E_\pi = X$, $\alpha\hat{}(1-i) \notin E_\pi = X$ implies $\alpha\hat{}i \in E_\pi = X$ so that $Con(X \cup \{\alpha\hat{}i\})$ holds. Thus we must assume that there is a largest m, say n, such that $(\pi(0),\ldots,\pi(m)) \in X$. Then rules of the form of (8) will allow us to show that $(\pi(0),\ldots,\pi(m)) \in X = cl_{mon}(X)$ for all $m \leq n$. Next rules of the form (6) will allow us to show that for all $m \leq n$, $\overline{(\pi(0),\ldots,\pi(m-1),1-\pi(m))} \in X = cl_{mon}(X)$ and then rules of the form (3) will allow us to show that $\overline{\alpha} \in X = cl_{mon}(X)$ for all $\alpha \in T$ such that α and η are incomparable where $\eta = (\pi(0),\ldots,\pi(n))$. Next we claim that it must be the case that both $\eta\hat{}0$ and $\eta\hat{}1$ lie on infinite paths through T. For if not, suppose $\eta\hat{}i$ does not lie on an infinite path through T. Then by König's Lemma $\{\beta : \beta \in T \,\&\, \beta \sqsupseteq \eta\hat{}i\}$ is finite and we can argue as above that repeated use of the rules in (2) and (4) will show $\overline{\eta\hat{}i} \in cl_{mon}(\emptyset) \subseteq X$. But then the rule $\frac{\eta,\overline{\eta\hat{}i}:}{\eta\hat{}(1-i)}$ from (9) shows $\eta\hat{}(1-i) \in cl_{mon}(X) = X$ violating our choice of η. Next suppose that $\eta\hat{}i$ does not lie on π and $\eta\hat{}(1-i)$ lies on π. Now consider the set $A = \{\alpha : \alpha \sqsupset \eta\hat{}i \text{ and } \overline{\alpha} \notin X\}$. Note that rules of the form (3) ensure that if $\alpha \in A$ and $\eta\hat{}i \sqsubseteq \beta \sqsubseteq \alpha$, then $\beta \in A$. That is, if $\beta \notin A$, then $\overline{\beta} \in X$ and hence $\overline{\alpha} \in cl_{mon}(X) = X$ by repeated use of the rules in (3). Thus the set of nodes in A determine a subtree of T rooted a η. We claim that there are no nodes $\beta \in A$ which are terminal with respect to A and hence A is infinite. For suppose $\beta \in A$ is a node which is terminal with respect to A. Thus $\overline{\beta} \notin X$ and hence β is not a terminal node of T because otherwise the rules of (2) would ensure $\overline{\beta} \in cl_{mon}(\emptyset) \subseteq X$. But then it must be the case that $\beta\hat{}0, \beta\hat{}1 \in T$ and $\overline{\beta\hat{}0}, \overline{\beta\hat{}1} \in X$. But then the rule $\frac{\overline{\beta\hat{}0},\overline{\beta\hat{}1}:}{\overline{\beta}}$ from (4) would show that $\overline{\beta} \in cl_{mon}(X) = X$ violating our choice of β. Thus A has no terminal nodes and hence A is infinite since $\eta\hat{}i \in A$. But this means that there is a least one path $\pi^* = (\pi^*(0), \pi^*(1), \ldots)$ through T such that $\eta\hat{}i$ is on π^* and $\overline{(\pi^*(0),\ldots,\pi^*(k))} \notin X$ for all k. Thus $X \subseteq E_\pi$ and $X \subseteq E_{\pi^*}$. Finally, it is easy to see that the only rules of the form $\frac{\alpha:\alpha\hat{}j}{\alpha\hat{}(1-j)}$ such that $\alpha \in X$, $\alpha\hat{}j \notin X$ are such that

1. (a) $\alpha \sqsubset \eta$ and $\alpha\hat{}(1-j) \sqsubseteq \eta$,
2. (b) $\alpha = \eta$, $j = i$, and $\alpha\hat{}(1-j) = \eta\hat{}(1-i)$, or
3. (c) $\alpha = \eta$, $j = 1-i$, and $\alpha\hat{}(1-j) = \eta\hat{}i$.

In cases (a) or (b), $\alpha\hat{}(i-j)$ is on the path π and hence $X \cup \{\alpha\hat{}(1-j)\} \subseteq E_\pi$ so that $Con(X \cup \{\alpha\hat{}(1-j)\})$ holds. In case (c), $\alpha\hat{}(1-j)$ is on the path π^* and $X \cup \{\alpha\hat{}(1-j)\} \subseteq E_{\pi^*}$ so that $Con(X \cup \{\alpha\hat{}(1-j)\})$ also holds in this case. Thus we have shown that all rules of nmon(S) are FC-normal with respect to Con. $\square$

Note that in the case where $[T]$ has no recursive elements,
$\{\eta \in T : \eta \text{ is an infinite path through } T\}$ is not recursive and hence we will not be

able to effectively decide whether a finite set $S \subseteq U$ is consistent. Thus it is crucial that we make no assumptions about the effectiveness of the consistency property.

9 Conclusions

In this paper we exhibited and investigated a natural condition which ensures that a logic program always has a stable model or a default theory or truth maintenance system always has an extension. This condition, called in our paper "FC-normality" is an abstraction of Reiter's notion of normality in default logic coupled with Scott's notion of Information System. This condition can be formulated in the general setting of nonmonotonic rules systems and hence can be immediately translated into other nonmonotonic reasoning formalisms including logic programming with negation as failure, default logic, and truth maintenance systems..

We have shown that FC-normal nonmonotonic rule systems (and, consequently. FC-normal logic programs, FC-normal logic programs with classical negation, FC-normal truth maintenance systems) have all the desirable properties of Reiter's normal default theories such as the existence of extensions, the semimontonicity property of extensions, and the orthogonality of extensions. Indeed, when FC-normal nonmonotonic rule systems are translated into the language of default logic, we get a class of default theories which we call extended normal default theories, which strictly contain the class of normal default theories of Reiter and yet continue to have all the desirable properties of Reiter's normal default theories.

We gave a general construction, which we called the forward chaining construction, which constucted all possible extensions of an FC-normal nonmonotonic rule system based on well orderings of the set of nonomontonic rules of the system. By analyzing the foward chaining construction, we were able to establish bounds on the complexity of reasoning with FC-ormal systems. For example, while there are recursive nonmonotonic rule systems which have extensions but which have no hyperarithmetic extensions, we showed that every recursive normal nonmonotonic rule systems has an extension which is r.e. in $0'$. Moreover, for any finite nonmonotonic rule system, we showed that one can always construct an extension in polynomial time.

We hope that the technique presented in this paper can serve as an indication to designers of intelligent systems, in particular to monitoring systems, as well as to logic programmers why some programs behave better than other programs. FC-normal programs offer a "smooth transition" from one belief state to another. In particular modification of belief rules (but not of "hard facts") can be handled smoothly by such systems. This is one way to maintain programs in a way which permits one to expand the present belief set to a larger one in the presence of new rules.

It is expected that a further research into the nature of consistency properties for nonmonotonic rule systems and their complexity will produce general techniques for writting and maintaining FC-normal nonmonotonic rule system (and hence FC-normal logic programs, extended normal default logics, and FC-normal truth maintenance systems) for specific applications.

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