A New Index for Polytopes

Margaret M. Bayer*
Department of Mathematics
University of Kansas
Lawrence, KS 66045

Andrew Klapper
College of Computer Science
Northeastern University
Boston, MA 02115

Abstract

A new index for convex polytopes is introduced. It is a vector whose length is the dimension of the linear span of the flag vectors of polytopes. The existence of this index is equivalent to the generalized Dehn-Sommerville equations. It can be computed via a shelling of the polytope. The ranks of the middle perversity intersection homology of the associated toric variety are computed from the index. This gives a proof of a result of Kalai on the relationship between the Betti numbers of a polytope and those of its dual.

1 Introduction

The combinatorial study of convex polytopes was invigorated by the proof of the characterization of face vectors of simplicial polytopes. The power and elegance of the techniques used in the simplicial case inspired attempts to adapt them to the study of arbitrary polytopes. This paper is the result of one such attempt. We start with a very brief history of the simplicial case, and then describe the analogy used here.

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McMullen [14] in 1970 conjectured that a certain set of conditions characterize the face vectors of simplicial polytopes. Various techniques were used to prove the necessity of certain subsets or weakenings of these conditions. The milestones were the Dehn-Sommerville equations (Sommerville, 1927 [15]), the Upper Bound Theorem (McMullen, 1970 [13], Stanley, 1975 [16]), and the Lower Bound Theorem (Barnette, 1973 [1]). Finally, in 1980, Stanley [18] proved the necessity of the McMullen conditions; the sufficiency was proved at the same time by Billera and Lee [3].

The various advances towards the proof of the McMullen conditions for simplicial polytopes proceeded from the discoveries of combinatorial interpretations for the h-vector (the image of the face vector under a certain linear transformation). The h-vector counts something in a shelling of the polytope [13], in the Stanley-Reisner ring of the polytope [16], and, finally, in the homology ring of the toric variety of the polytope [18].

In 1983 the affine span of the flag vectors of arbitrary polytopes was determined [2]. At about the same time Stanley introduced to combinatorists a formula for the "generalized h-vector", giving the ranks of the middle perversity intersection homology of the toric variety of an arbitrary polytope [19]. The analogy with the simplicial case motivated us to search for a connection between flag vectors and generalized h-vectors. The generalized h-vector is too small to incorporate all the flag vector information. The flag vector has no nice interpretation in terms of shellings or rings. Thus we wanted some extension of the generalized h-vector or transformation of the flag vector that could be used easily both to describe the linear relations on flag vectors and to find the homology ranks for the toric variety. A connection with shellings of polytopes was also desirable.

Kalai [11] gives one solution to this problem. Another approach uses the generalized h-vectors of "relative" posets of the polytope [19]. Neither of these has a shelling interpretation, however. Here we present a different approach, by introducing the "cd index" of a polytope. This index encapsulates the linear relations on flag vectors. It can be computed from a shelling of the polytope. The homology ranks can be calculated from this index. This calculation enables us to prove a result conjectured by Kalai (who has found another proof as well [10]).

Beyond their intrinsic interest, the results in this paper are significant as an example of the use of mathematical experimentation by computer. High speed computation allowed us to examine enough examples to make evident the underlying patterns. More discussion of the computation follows the proof of Theorem 9 in Section 5. The authors hope that our success with this problem will inspire others to use automated generation of mathematical examples.

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2 Definitions

Though we are interested primarily in convex polytopes, some of the proofs are simplified by considering the broader class of regular CW spheres. A set in $\mathbf{R}^n$ is an *open k-cell* if it is homeomorphic to the interior of the k-dimensional unit ball for some k; an open cell is *regular* if its closure is homeomorphic to the closed unit ball. A (finite) *regular CW complex* is a closed subset of $\mathbf{R}^n$ that is partitioned into a finite number of regular open cells, each of whose closures is the union of some of these open cells. A *regular CW sphere* is a regular CW complex that is homeomorphic to a sphere. A *polytope* is the convex hull of a finite point set in $\mathbf{R}^n$. We often use the term polytope when we really mean its boundary, which is a regular CW sphere.

The closure of a regular open k-cell is called a (k-dimensional) *face* of the complex. Following [4] we call a finite regular CW complex an $(n-1)$-*CW-complex* if every face is contained in some $(n-1)$-face; the $(n-1)$-faces are called *facets*. An $(n-1)$-CW-complex P (e.g., an n-polytope) has proper faces of dimension zero through $n-1$; the empty set is considered an improper face of dimension -1, with P itself an improper face of dimension n. The faces, ordered by inclusion, form the face poset of the regular CW complex P (with least element $\emptyset$ and greatest element P), and this poset is a lattice when the complex is a polytope. We are primarily interested in these posets for regular CW spheres, and will blur the distinction between a sphere and its face poset.

For P an $(n-1)$-CW-complex let $f_i(P)$ be the number of i-dimensional faces of P, and let the *f-vector* of P be $f(P) = (f_0(P), f_1(P), \ldots, f_{n-1}(P))$. (For general information on polytopes and f-vectors, see [9].) A chain of faces $\emptyset \subset F_1 \subset F_2 \cdots \subset F_k \subset P$ is called an *S-flag*, where $S = \{\dim F_i : 1 \leq i \leq k\}$. Let $f_S(P)$ be the number of S-flags of P, and let the *flag vector* or extended f-vector of P be $(f_S(P))_{S \subseteq \{0,1,\ldots,n-1\}} \in \mathbf{N}^{2^n}$.

Stanley [17] introduced a transformation of the flag vector, and interpreted the new vector in terms of a shelling of the barycentric subdivision of the sphere. This vector will be denoted by β here, instead of h, as used in [17] and [2]. Since those papers, h has come to be used for the intersection homology Betti numbers, which will be discussed in Section 4. The letter β has been used in poset theory for the vector we define here.

Definition. Suppose $(f_S(P))_{S \subseteq \{0,1,\ldots,n-1\}} \in \mathbf{N}^{2^n}$ is the flag vector of a regular CW complex P. The *β-vector* of P is the vector $(\beta_S(P))_{S \subseteq \{0,1,\ldots,n-1\}} \in \mathbf{N}^{2^n}$ given by

$$\beta_S(P) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} f_T(P).$$

This transformation is invertible:

$$f_S(P) = \sum_{T \subseteq S} \beta_T(P).$$

We use a generating function in the algebra $\mathbf{Q}\langle a, b\rangle$ of polynomials in the noncommuting variables a and b. For $S \subseteq \{0, 1, \ldots, n-1\}$ write $w_i = a$ if $i \notin S$ and $w_i = b$ if $i \in S$; let $w_S = w_0 w_1 \ldots w_{n-1}$. The generating function for the β-vector is then

$$\beta(P) = \sum_{S \subseteq \{0,1,\ldots,n-1\}} \beta_S(P) w_S.$$

Definitions of shellings vary a bit. The following one (from [4]) applies to regular CW complexes. For σ a k-face denote by $\partial\sigma$ the boundary of σ, a $(k-1)$-sphere.

Definition. An ordering $\sigma_1, \sigma_2, \ldots, \sigma_t$ of the facets of an $(n-1)$-CW-complex is a *shelling* if $n = 1$ or if $n > 1$ and

i. $\partial\sigma_1$ has a shelling;

ii. $\sigma_j \cap (\cup_{i=1}^{j-1} \sigma_i)$ is an $(n-2)$-CW-complex, $j = 2, 3, \ldots, t$; and

iii. $\partial\sigma_j$ has a shelling in which the $(n-2)$-faces of $\sigma_j \cap (\cup_{i=1}^{j-1} \sigma_i)$ come first, $j = 2, 3, \ldots, t$.

A shelling $\sigma_1, \sigma_2, \ldots, \sigma_t$ of a regular CW complex is *reversible* if $\sigma_t, \sigma_{t-1}, \ldots, \sigma_1$ is also a shelling. A regular CW complex is *shellable* if it has a shelling. All polytopes have reversible shellings. Not all regular CW spheres are shellable.

Stanley's computation of the β-vector shows that the $\beta_S(P)$ are nonnegative, and that for all $S \subseteq \{0, 1, \ldots, n-1\}$, $\beta_S(P) = \beta_{\tilde{S}}(P)$, where $\tilde{S}$ is the complement of S. These are not all linear equations satisfied by the β-vectors of regular CW spheres. (Stanley was studying a broader class of objects.) A complete set, called the *generalized Dehn-Sommerville equations*, is derived in [2]. The exact form of the equations is not important to us now, but we will need to know the dimension of the space they define.

Theorem 1 *The dimension of the linear span of the β-vectors of regular CW $(n-1)$-spheres is e_n, where (e_n) is the Fibonacci sequence initialized by $e_0 = e_1 = 1$.*

Our proofs will be facilitated by two constructions on regular CW spheres. Let Q be a shellable regular CW $(n-1)$-sphere. Let XQ be the regular CW n-sphere obtained by attaching two n-cells, ρ_1 and ρ_2, each with boundary Q. Let YQ be the regular CW $(n+1)$-sphere obtained by attaching to Q three n-cells, ρ_1, ρ_2 and ρ_3, each with boundary Q, and then attaching three

$(n+1)$-cells, τ_1, τ_2 and τ_3, each having two of ρ_1, ρ_2 and ρ_3 in its boundary. Denote by $\mathcal{C}$ be the set of all regular CW spheres obtained by starting with $\emptyset$ and applying the X and Y constructions any number of times. Write $\mathcal{C}_n$ for the set of $(n-1)$-dimensional spheres in $\mathcal{C}$.

3 The *cd* index and the generalized Dehn-Sommerville equations

In this section we give a recursive formula for the β-vector of a regular CW sphere and derive the *cd* index. Fix a regular CW sphere P and a reversible shelling $\sigma_1, \sigma_2, \ldots, \sigma_t$. For each j, $1 \leq j \leq t$, let $U_j = \sigma_j \cap (\cup_{i<j} \sigma_i)$, $L_j = \sigma_j \cap (\cup_{i>j} \sigma_i)$, and $E_j = U_j \cap L_j$. For $2 \leq j \leq t-1$, U_j and L_j are $(n-2)$-balls, and E_j is an $(n-3)$-sphere; $U_1 = L_t = E_1 = E_t = \emptyset$, $L_1 = \partial\sigma_1$ and $U_t = \partial\sigma_t$. Theorem 2 gives the generating function $\beta(P)$ in terms of $\beta(\partial\sigma_j)$ and $\beta(E_j)$.

Theorem 2 *Let P be a regular CW sphere with reversible shelling $\sigma_1, \sigma_2, \ldots, \sigma_t$, and let U_j, L_j and E_j be defined as above. Then*

i. $\beta(\partial\sigma_j) = \beta(U_j) + \beta(L_j) + \beta(E_j)(b-a)$, *for* $1 \leq j \leq t$.

ii. $\beta(P) = \displaystyle\sum_{j=1}^{t} \beta(\partial\sigma_j)a + \beta(U_j)(b-a)$.

iii. $\beta(P) = \displaystyle\frac{1}{2}\sum_{j=1}^{t} \beta(\partial\sigma_j)(a+b) + \sum_{j=2}^{t-1} \beta(E_j)\left((ab+ba) - \frac{1}{2}(a+b)(a+b)\right)$.

Proof. These are straightforward calculations in terms of the flags of the sphere P.

i. We have $\beta_S(\partial\sigma_j) = \beta_S(U_j) + \beta_S(L_j) - \beta_S(E_j)$. Observe that if a flag is contained in both U_j and L_j (that is, in E_j), then its top element has dimension at most $n-3$. If $n-2 \in S$ then

$$\beta_S(E_j) \quad = \quad \sum_{T \subseteq S} (-1)^{|S \setminus T|} f_T(E_j)$$

$$= \sum_{T\subseteq S\setminus\{n-2\}} (-1)^{|S\setminus T|} f_T(E_j) = -\beta_{S\setminus\{n-2\}}(E_j)$$

In the generating function we get equation i.

ii. We show that $\beta(\cup_{i\leq j}\sigma_i) = \beta(\cup_{i<j}\sigma_i) + \beta(\partial\sigma_j)a + \beta(U_j)(b-a)$. For each S, $\beta_S(\cup_{i\leq j}\sigma_i) = \beta_S(\cup_{i<j}\sigma_i) + \beta_S(\sigma_j) - \beta_S(U_j)$. If $n-1 \notin S$, then $\beta_S(\sigma_j) = \beta_S(\partial\sigma_j)$, so $\beta_S(\cup_{i\leq j}\sigma_i) = \beta_S(\cup_{i<j}\sigma_i) + \beta_S(\partial\sigma_j) - \beta_S(U_j)$. If $n-1 \in S$, then in $\beta_S(\sigma_j) = \sum_{T\subseteq S}(-1)^{|S\setminus T|} f_T(\sigma_j)$, each flag of proper faces of σ_j cancels with a flag ending in σ_j itself, so $\beta_S(\sigma_j) = 0$. For this case (as in the calculation of $\beta_S(E_j)$ in part i) $\beta_S(U_j) = -\beta_{S\setminus\{n-1\}}(U_j)$. So for $n-1 \in S$, $\beta_S(\cup_{i\leq j}\sigma_i) = \beta_S(\cup_{i<j}\sigma_i) + \beta_{S\setminus\{n-1\}}(U_j)$. In the generating function we get $\beta(\cup_{i\leq j}\sigma_i) = \beta(\cup_{i<j}\sigma_i) + \beta(\partial\sigma_j)a + \beta(U_j)(b-a)$. So equation ii holds by induction.

iii. When the shelling is reversed the U_j and L_j are exchanged and the E_j stay the same. Adding the resulting version of equation ii to the original and applying equation i give

$$\begin{aligned} 2\beta(P) &= \sum_{j=1}^{t} 2\beta(\partial\sigma_j)a + (\beta(U_j) + \beta(L_j))\,(b-a) \\ &= \sum_{j=1}^{t} 2\beta(\partial\sigma_j)a + (\beta(\partial\sigma_j) - \beta(E_j)(b-a))\,(b-a) \\ &= \sum_{j=1}^{t} \beta(\partial\sigma_j)(a+b) + \beta(E_j)\,(2(ab+ba) - (a+b)(a+b))\,. \quad \square \end{aligned}$$

According to Stanley's computation of the β-vector [17], $\beta(P) \in \mathbf{Q}\langle a,b\rangle$ is symmetric under the action that exchanges a and b. Theorem 2, combined with initial conditions, gives a stronger condition on $\beta(P)$: it is in the subalgebra generated by two elements, $a+b$ and $ab+ba$. We abbreviate these two elements as $c = a+b$, $d = ab+ba$, and write $\mathbf{Q}\langle c,d\rangle$ for the subalgebra they generate. For every regular CW sphere with a reversible shelling the β-vector has a generating function in $\mathbf{Q}\langle c,d\rangle$. This is called the *cd index* of the sphere; it is written $\sum_w \beta_w w$, where the sum is over all words w in c and d. The *cd* index is given recursively in the following corollary.

Corollary 3 *Let P be a regular CW $(n-1)$-sphere with reversible shelling $\sigma_1, \sigma_2, \ldots, \sigma_t$. For $2 \le i \le t-1$, let $E_i = \sigma_i \cap (\cup_{j<i}\sigma_j) \cap (\cup_{j>i}\sigma_j)$.*

i. If $n = 1$, then $\beta(P) = c$.

ii. If $n = 2$ and P has k vertices, then $\beta(P) = cc + (k-2)d$.

iii. If $n \ge 3$, then $\beta(P) = \frac{1}{2}\sum_{j=1}^{t} \beta(\partial\sigma_j)c + \sum_{j=2}^{t-1} \beta(E_j)(d - \frac{1}{2}cc)$.

In fact the *cd* index exists even for nonshellable spheres and, more generally, for all Eulerian posets (for the definition see [2]). That is, as we shall see, the *cd* index exists wherever the generalized Dehn-Sommerville equations hold. Define the *weight* of a word in c and d to be the degree of its expansion as a homogeneous polynomial in a and b. Thus if c occurs i times and d occurs j times in w, then the weight of w is $i + 2j$.

We now compute the effect on the *cd* index of the X and Y constructions on spheres. For both XQ and YQ any ordering of the facets (ρ_i for XQ, τ_i for YQ) is a shelling, so Corollary 3 applies to give

$$\begin{aligned}
\beta(XQ) &= 1/2\,(\beta(\partial\rho_1) + \beta(\partial\rho_2))\,c = \beta(Q)c \\
\beta(YQ) &= 1/2\,(\beta(\partial\tau_1) + \beta(\partial\tau_2) + \beta(\partial\tau_3))\,c + \beta(\rho_1 \cap \rho_3)(d - 1/2cc) \\
&= 3/2\beta(XQ)c + \beta(Q)(d - 1/2cc) = \beta(Q)(cc + d).
\end{aligned}$$

The following was first observed in discussions with Fine (see also [7]).

Theorem 4 *Let P be a graded poset. Then P has a cd index with integer coefficients if and only if the β-vector of P satisfies the generalized Dehn-Sommerville equations.*

Proof. Let $\mathcal{E}$ be the subspace of $\mathbf{Q}\langle a, b\rangle$ generated by (the generating functions of) β-vectors satisfying the generalized Dehn-Sommerville equations. We wish to show that $\mathbf{Q}\langle c, d\rangle = \mathcal{E}$. First note that the two are graded vector spaces with components the homogeneous polynomials in a and b of fixed degree. Theorem 1 says that the dimension of the degree n component of $\mathcal{E}$ is e_n (Fibonacci number). It is the same for $\mathbf{Q}\langle c,d\rangle$. To see this observe that there is no algebra relation in $\mathbf{Q}\langle a,b\rangle$ between $a+b$ and $ab+ba$ (see, for example, [5]). So the dimension of the degree n component of $\mathbf{Q}\langle c,d\rangle$ is the number of cd words of weight n. An inductive combinatorial argument easily shows this number is e_n.

We now give a subset of $\mathcal{E}$ that is a basis for $\mathbf{Q}\langle c,d\rangle$. Let $\beta(\mathcal{C}) \subset \mathcal{E}$ be the set of polynomials $\beta(P)$ (including $\beta(\emptyset)=1$) for the spheres P in $\mathcal{C}$, the spheres obtained by using the X and Y constructions. We show by induction on degree that $\beta(\mathcal{C})$ spans $\mathbf{Q}\langle c,d\rangle$ as a vector space. Suppose w is a degree n element of $\mathbf{Q}\langle c,d\rangle$, and $w=\sum_{Q\in\mathcal{C}_n}\alpha_Q\beta(Q)$ for some $\alpha_Q\in\mathbf{Q}$. Then

$$wc = \sum_{Q\in\mathcal{C}_n}\alpha_Q\beta(Q)c = \sum_{Q\in\mathcal{C}_n}\alpha_Q\beta(XQ)$$

and

$$\begin{aligned} wd = \sum_{Q\in\mathcal{C}_n}\alpha_Q\beta(Q)d &= \sum_{Q\in\mathcal{C}_n}\alpha_Q\beta(Q)(cc+d-cc) \\ &= \sum_{Q\in\mathcal{C}_n}\alpha_Q\left(\beta(YQ)-\beta(XXQ)\right). \end{aligned}$$

Since $1=\beta(\emptyset)$, all cd words can be written as linear combinations of elements of $\beta(\mathcal{C})$. The same combinatorial argument referred to above shows that $\mathcal{C}_n$ has e_n elements. Thus $\beta(\mathcal{C})$ is a vector space basis for $\mathbf{Q}\langle c,d\rangle$, so $\mathbf{Q}\langle c,d\rangle=\mathcal{E}$.

It remains to show that the coefficients of the cd index are integers; this is not obvious from Corollary 3. Let M_n be the $2^n\times e_n$ matrix that expands homogeneous cd polynomials of degree n out as ab polynomials. For a cd word w, let $\psi(w)$ be the lexicographically first ab word occurring in w. The word $\psi(w)$ is obtained by replacing each c by a and each d by ab; the set of words arising in this way are those beginning with a and having no two consecutive bs. If v occurs before w in lexicographic order, then $\psi(v)$ occurs before $\psi(w)$ in lexicographic order. Thus the rows of M_n indexed by the words $\psi(w)$ form an $e_n\times e_n$ lower triangular submatrix of M_n with $0,1$ entries and ones along the diagonal. So any polynomial in $\mathbf{Z}\langle a,b\rangle$ known to be in $\mathbf{Q}\langle c,d\rangle$ can be written as an integer combination of cd words. $\square$

Thus the *cd* index of a regular CW sphere encodes the numbers of flags in the most efficient way, i.e., without any redundancy reflected in the generalized Dehn-Sommerville equations. Here are the *cd* indices of low dimensional spheres in terms of the flag vectors. Note that set brackets have been omitted from the flag numbers and 1 is written instead of $f_\emptyset$.

n=2 $\beta(P) = cc + (f_1 - 2)dc$

n=3 $\beta(P) = ccc + (f_2 - 2)cd + (f_1 - f_2)dc$

n=4 $$\beta(P) = cccc + (f_3 - 2)ccd + (f_2 - f_3)cdc + (f_1 - f_2 + f_3 - 2)dcc + (f_{13} - 2f_1 - 2f_3 + 4)dd$$

The *cd* index also inherits a nice property from the flag vector. Let P be a regular CW sphere with a dual sphere P^*; that is, the face poset of P^* is the dual of the face poset of P. For $S = \{i_1, i_2, \ldots, i_k\} \subseteq \{0, 1, \ldots, n-1\}$, write $S^* = \{n-1-i_k, n-1-i_{k-1}, \ldots, n-1-i_1\}$. Then the flag vector and β-vector of P^* are related to those of P by $f_S(P^*) = f_{S^*}(P)$ and $\beta_S(P^*) = \beta_{S^*}(P)$. This implies that the *cd* index of P^* is obtained from the *cd* index of P by reversing each *cd* word.

Fine made the following conjecture for polytopes. It holds for regular CW spheres of dimension three or less.

Conjecture 5 *The coefficients in the cd index of a regular CW sphere are nonnegative.*

4 The generalized h-vector

Much of the study of simplicial polytopes has used the "h-vector" of the polytope, obtained from the f-vector by a certain linear transformation. In particular, Stanley's proof of the necessity of the McMullen conditions for the f-vector of a simplicial polytope [18] depends on the fact that the h-vector gives the ranks of the homology groups of the "toric variety" associated with the polytope. Applying the h-vector transformation to the f-vector of a nonsimplicial polytope does not give anything meaningful. Furthermore, the ordinary homology Betti numbers of the toric variety associated with an arbitrary polytope are not determined by the face lattice [12]. However, several algebraic geometers (J. N. Bernstein, A. G. Khovanskii and R. D. MacPherson) independently developed formulas for the (middle perversity) intersection homology Betti numbers of the varieties associated with arbitrary (not necessarily simplicial) rational polytopes. For relatively accessible references on the algebraic geometry see [6, 8, 19, 20]. In [19] Stanley generalized these Betti number formulas to Eulerian posets and studied the resulting "generalized h-vector" for polytopes. We present the definition in the context of regular CW spheres.

Definition. Every regular CW $(n-1)$-sphere P has a *generalized h-vector* $(h_0, h_1, \ldots, h_n) \in \mathbf{N}^{n+1}$, with generating function $h(P,t) = \sum_{i=0}^{n} h_i t^i$, and *$g$-vector* $(g_0, g_1, \ldots, g_{\lfloor n/2 \rfloor}) \in \mathbf{N}^{n+1}$, with generating function $g(P,t) = \sum_{i=0}^{\lfloor n/2 \rfloor} g_i t^i$, related by $g_0 = h_0$ and $g_i = h_i - h_{i-1}$ for $1 \leq i \leq \lfloor n/2 \rfloor$. The generalized h-vector and g-vector are defined by the recursion

i. $g(\emptyset, t) = h(\emptyset, t) = 1$, and

ii. $h(P,t) = \displaystyle\sum_{\substack{G \text{ face of } P \\ G \neq P}} g(\partial G, t)(t-1)^{n-1-\dim G}$.

When P is a simplicial n-polytope, this definition gives the h-vector used in the study of simplicial polytopes; hence it satisfies the McMullen conditions. The g-vector of any regular CW sphere determines the generalized h-vector, because the generalized h-vector is symmetric [19]. For rational polytopes the algebraic geometry shows that the generalized h-vector is nonnegative and unimodal, but it is not known whether the other McMullen conditions hold. Nonnegativity fails for some regular CW spheres, however. The h-vector of a simplicial polytope depends linearly on the f-vector. In general, the generalized h-vector depends on the flag vector.

Theorem 6 *For any regular CW $(n-1)$-sphere and any i, $0 \leq i \leq n$, h_i is an integer linear combination of f_S, as S ranges over subsets of $\{0, 1, \ldots, n-1\}$.*

Proof. The proof is by induction on n. Clearly for fixed n if the h_i are linear functions of the f_S, then the g_i are also. For $n = 1$ the proposition holds (using $f_\emptyset = 1$). Assume the proposition is true for dimensions less than $n-1$. Then for an $(n-1)$-sphere P, $h(P,t)$ is the sum, over faces of dimension less than n, of polynomials whose coefficients are integer linear combinations of the flag numbers of the faces. The linear combinations themselves depend only on the dimension of the face. So for some integers $a_{k,i,S}$

$$\begin{aligned} h(P,t) &= \sum_{k=-1}^{n-1} \sum_{\substack{G \text{ face of } P \\ \dim G = k}} \sum_{i=0}^{n} \left(\sum_{S \subseteq \{0,1,\ldots,k-1\}} a_{k,i,S} f_S(G) \right) t^i \\ &= \sum_{k=-1}^{n-1} \sum_{i=0}^{n} \sum_{S \subseteq \{0,1,\ldots,k-1\}} a_{k,i,S} \left(\sum_{\substack{G \text{ face of } P \\ \dim G = k}} f_S(G) \right) t^i. \end{aligned}$$

But for $S \subseteq \{0,1,\ldots,k-1\}$, $\displaystyle\sum_{\substack{G \text{ face of } P \\ \dim G = k}} f_S(G) = f_{S\cup\{k\}}(P)$. So $h(P,t) = \displaystyle\sum_{k=-1}^{n-1}\sum_{i=0}^{n}\sum_{S \subseteq \{0,1,\ldots,k-1\}} a_{k,i,S} f_{S\cup\{k\}}(P) t^i$, a polynomial whose coefficients, $h_0, h_1, \ldots, h_n$, are integer linear combinations of f_T, as T ranges over subsets of $\{0,1,\ldots,n-1\}$. $\square$

Thus the flag vector, and hence the *cd* index, of a regular CW sphere determines its generalized h-vector, but the reverse is clearly not the case. Kalai [11] has found a set of parameters, containing the generalized h-vector, which determines the flag vector and is nonnegative for rational polytopes.

The generalized h-vector of a regular CW sphere does not by itself determine the generalized h-vector of a dual sphere. Kalai [10] conjectured the following relation holds. He later found a proof (unpublished) of this conjecture; in the next section we show how this result also follows from the calculation of the generalized h-vector in terms of the *cd* index.

Theorem 7 *For n even and P any n-polytope, $g_{n/2}(P) = g_{n/2}(P^*)$.*

5 The generalized h-vector in terms of the cd index

In this section we compute the generalized h-vector of a sphere in terms of its *cd* index. First we calculate the effect on the generalized h-vector of the X and Y constructions.

Proposition 8 *For any regular* CW $(n-1)$*-sphere* Q

i. $\beta(XQ) = \beta(Q)c$

ii. $\beta(YQ) = \beta(Q)(cc+d)$

iii. $h_i(XQ) = \begin{cases} h_i(Q) - h_{i-1}(Q), & \text{if } 0 \leq i \leq \frac{n}{2} \\ h_{i-1}(Q) - h_i(Q), & \text{if } \frac{n}{2} < i \leq n+1 \end{cases}$

iv. $h_i(YQ) - h_i(XXQ) =$

$$\begin{cases} h_{i-1}(Q) - h_{i-2}(Q), & \text{if } n \text{ is even and } i = (n+2)/2 \\ 0, & \text{otherwise} \end{cases}$$

Proof. Equations i and ii were derived in Section 3. For equation iii recall that the proper faces of XQ are the proper faces of Q and two copies of Q itself. Thus

$$\begin{aligned} h(XQ,t) &= \sum_{\substack{G \text{ face of } XQ \\ G \neq XQ}} g(\partial G,t)(t-1)^{n-\dim G} \\ &= (t-1) \sum_{\substack{G \text{ face of } Q \\ G \neq Q}} g(\partial G,t)(t-1)^{n-1-\dim G} + 2g(Q,t) \\ &= (t-1)h(Q,t) + 2g(Q,t). \end{aligned}$$

So with the convention $h_{-1}(Q) = h_{n+1}(Q) = 0$,

$$\sum_{i=0}^{n+1} h_i(XQ)t^i = \sum_{i=0}^{n+1} \left(h_{i-1}(Q) - h_i(Q)\right) t^i + 2 \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \left(h_i(Q) - h_{i-1}(Q)\right) t^i.$$

This gives equation iii.

For equation iv note that the faces of YQ are all isomorphic to faces of XXQ; the difference between the two face lattices is that YQ has one more face of type Q and one more face of type XQ. Thus

$$\begin{aligned} h(YQ,t) - h(XXQ,t) &= \sum_{\substack{G \text{ face of } YQ \\ G \text{ not a face of } XXQ \\ G \neq YQ}} g(\partial G, t)(t-1)^{n+1-\dim G} \\ &= (t-1)g(Q,t) + g(XQ,t). \end{aligned}$$

This is a polynomial of degree at most $\lfloor \frac{n}{2} \rfloor + 1$. For $0 \le i \le \lfloor \frac{n}{2} \rfloor$ $h_i(XQ) = g_i(Q)$, so the coefficient of t^i is

$$h_i(YQ) - h_i(XXQ) = g_{i-1}(Q) - g_i(Q) + h_i(XQ) - h_{i-1}(XQ) = 0.$$

If n is even, $g(XQ,t)$ has degree at most $n/2$, so $h(YQ,t) - h(XXQ,t) = g_{n/2}(Q)t^{\frac{n}{2}+1}$. If n is odd, $h_{(n+1)/2}(Q) = h_{(n-1)/2}(Q)$, so the coefficient of $t^{(n+1)/2}$ in $g(XQ,t)$ is

$$\begin{aligned} g_{(n+1)/2}(XQ) &= h_{(n+1)/2}(XQ) - h_{(n-1)/2}(XQ) \\ &= \big(h_{(n-1)/2}(Q) - h_{(n+1)/2}(Q)\big) \\ &\qquad - \big(h_{(n-1)/2}(Q) - h_{(n-3)/2}(Q)\big) = -g_{(n-1)/2}(Q). \end{aligned}$$

Therefore the coefficient of $t^{(n+1)/2}$ in $g(XQ,t)$ cancels the coefficient of $t^{(n+1)/2}$ in $(t-1)g(Q,t)$. So for n odd $h(YQ,t) - h(XXQ,t) = 0$. $\square$

Recall that the generalized h-vector is obtained from the flag vector (and hence from the *cd* index) by a linear transformation. Proposition 8 says that the matrix of this transformation is determined by the h-vectors and *cd* indices of spheres in the set $\mathcal{C}$, i.e., the spheres obtained by successively applying the X and Y constructions. Write β for the vector of coefficients β_w of the *cd* index. Let A_n be the transformation matrix for degree n; so $A_n \in \mathbf{Q}^{(n+1)\times e_n}$ and $h^T = A_n\beta^T$. The rows of this matrix are indexed by the set $\{0, 1, \dots, n\}$, and the columns are indexed by *cd* words of weight n. For w a *cd* word of weight n, write A^w for the column of A_n indexed by w, and $a_{i,w}$ for the ith element of this column.

Theorem 9 *Let w be a word of weight n, $w = c^{i_0}d^{j_0}c^{i_1}\cdots d^{j_{k-1}}c^{i_k}$.*

i. If $k = 0$ and $i_0 = n \geq 1$, then

$$A^w = A^{c^n} = \left[\begin{array}{c} a_{0,c^{n-1}} \\ a_{1,c^{n-1}} - a_{0,c^{n-1}} \\ \vdots \\ a_{\lfloor(n-1)/2\rfloor,c^{n-1}} - a_{\lfloor(n-3)/2\rfloor,c^{n-1}} \\ a_{\lfloor(n-1)/2\rfloor,c^{n-1}} - a_{\lfloor(n+1)/2\rfloor,c^{n-1}} \\ \vdots \\ a_{n-2,c^{n-1}} - a_{n-1,c^{n-1}} \\ a_{n-1,c^{n-1}} \end{array}\right]$$

where $A^{c^0} = [1]$.

ii. If $k \geq 1$ and for some r, $0 \leq r \leq k-1$, i_r is odd, then $A^w = 0$.

iii. If $k \geq 1$ and for all r, $0 \leq r \leq k-1$, i_r is even, then

$$A^w = \prod_{r=0}^{k-1} p_{i_r} \left[\begin{array}{c} 0^{(k)} \\ A^{c^{i_k}} \\ 0^{(k)} \end{array}\right]$$

where $0^{(k)}$ is a column of $(n-i_k)/2$ 0s, and for any even i, $p_i = a_{i/2,c} - a_{(i-2)/2,c}$.

Proof. For Q a regular CW $(m-1)$-sphere, Proposition 8 part i gives $[h(XQ)]^T = A_n[\beta(XQ)]^T = \sum_v A^{vc}\beta_v(Q)$. Applying part iii of the same Proposition, we get

$$\sum_v a_{i,vc}\beta_v(Q) = \begin{cases} \sum_v (a_{i,v} - a_{i-1,v})\beta_v(Q), & \text{if } 0 \leq i \leq m/2 \\ \sum_v (a_{i-1,v} - a_{i,v})\beta_v(Q), & \text{if } m/2 < i \leq m+1 \end{cases}.$$

The $\beta_v(Q)$ are independent functions of Q, so for all words v of weight m,

$$a_{i,vc} = \begin{cases} a_{i,v} - a_{i-1,v}, & \text{if } 0 \leq i \leq m/2 \\ a_{i-1,v} - a_{i,v}, & \text{if } m/2 < i \leq m+1 \end{cases}. \tag{1}$$

Similarly, for all words v of weight m,

$$a_{i,vd} = \begin{cases} a_{i-1,v} - a_{i-2,v}, & \text{if } m \text{ is even and } i = (m+2)/2 \\ 0, & \text{otherwise} \end{cases} . \tag{2}$$

Applying equation (1) with $v = c^{n-1}$ gives part i. By equation (2), if v is any word of odd length, and w is any word beginning with vd then $A_w = 0$. This gives part ii.

Finally we prove part iii by induction on w. Assume part iii holds for $v = c^{i_0} d^{j_0} c^{i_1} \cdots d^{j_{k-1}} c^{i_k - 1}$; we show it holds for vc. Equation (1) says that $A^{vc} = \partial(A^v)$, where ∂ is the "difference operator," $\partial(x_0, x_1, \ldots, x_m)^T = (x_0, x_1 - x_0, \ldots, x_{\lfloor (m-1)/2 \rfloor} - x_{\lfloor (m-3)/2 \rfloor}, x_{\lfloor (m-1)/2 \rfloor} - x_{\lfloor (m+1)/2 \rfloor}, \ldots, x_{m-1} - x_m, x_m)^T \in \mathbf{Q}^{m+2}$. Clearly for a scalar λ, $\partial(\lambda x) = \lambda \partial(x)$. Also, if $x' = (0, x, 0)^T$ then $\partial(x') = (0, \partial(x), 0)^T$. So

$$A^{vc} = \partial(A^v) = \prod_{r=0}^{k-1} p_{i_r} \begin{bmatrix} 0^{(k)} \\ \partial(A^{c^{i_k - 1}}) \\ 0^{(k)} \end{bmatrix} = \prod_{r=0}^{k-1} p_{i_r} \begin{bmatrix} 0^{(k)} \\ A^{c^{i_k}} \\ 0^{(k)} \end{bmatrix}.$$

So part iii holds for vc.

It remains to show that part iii holds for vd if it holds for v, and if each string of cs in v is even. Consider first the case $v = c^{i_0} d^{j_0} c^{i_1} \cdots d^{j_{k-2}} c^{i_{k-1}}$; note the weight of v is even. Equation (2) says that $A^{vd} = \gamma(A^v)$, where for m even $\gamma(x_0, x_1, \ldots, x_m)^T = (0, \ldots, 0, x_{m/2} - x_{m/2-1}, 0, \ldots, 0)^T \in \mathbf{Q}^{m+3}$. As before, for a scalar λ, $\gamma(\lambda x) = \lambda \gamma(x)$, and for $x' = (0, x, 0)^T$, $\gamma(x') = (0, \gamma(x), 0)^T$. So

$$\begin{aligned} A^{vd} = \gamma(A^v) &= \prod_{r=0}^{k-2} p_{i_r} \begin{bmatrix} 0^{(k-1)} \\ \gamma(A^{c^{i_{k-1}}}) \\ 0^{(k-1)} \end{bmatrix} = \prod_{r=0}^{k-2} p_{i_r} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ p_{i_{k-1}} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\ &= \prod_{r=0}^{k-1} p_{i_r} \begin{bmatrix} 0^{(k)} \\ 1 \\ 0^{(k)} \end{bmatrix} = \prod_{r=0}^{k-1} p_{i_r} \begin{bmatrix} 0^{(k)} \\ A^{c^0} \\ 0^{(k)} \end{bmatrix}. \end{aligned}$$

So for w ending in cd, part iii holds. Now if $w = vd$, with v itself ending in d, then part iii for v along with equation (2) show that $A^{vd} = [0\, A^v\, 0]^T$. So part iii holds in this case as well. $\square$

Theorem 9 was discovered by computing the matrix A_n for small weights n, using the relations in Proposition 8. As the weight increases, the basis of cd words grows exponentially, and the time required to compute the matrix soon becomes prohibitive. The formulas in Theorem 9 became evident only by comparing the generalized h-vectors corresponding to cd words of weight six, seven and eight. The computation was carried out on a VAX 11/750 in the Pascal programming language using dynamic programming techniques. Two hours of cpu time were required for the computation for cd words of weight eight. To perform this calculation by hand would require a tremendous amount of time, far more than the time it took to write the program. Moreover, we feel that in such hand calculations the possibility of undiscovered errors is much greater than in the machine calculation we performed.

As a corollary we get Kalai's result on dual spheres.

Corollary 10 *For n even and P any $(n-1)$-sphere with dual sphere P^*, $g_{n/2}(P^*) = g_{n/2}(P)$.*

Proof. It suffices to show that when n is even

$$a_{n/2,w} - a_{(n-2)/2,w} = a_{n/2,\bar{w}} - a_{(n-2)/2,\bar{w}} \tag{3}$$

for all cd words w of weight n, where $\overline{w}$ is the reverse of the word w. For then

$$\begin{aligned} g_{n/2}(P^*) &= h_{n/2}(P^*) - h_{(n-2)/2}(P^*) \\ &= \sum_w (a_{n/2,w} - a_{(n-2)/2,w})\beta_w(P^*) = \sum_w (a_{n/2,\bar{w}} - a_{(n-2)/2,\bar{w}})\beta_{\bar{w}}(P) \\ &= \sum_w (a_{n/2,w} - a_{(n-2)/2,w})\beta_w(P) = g_{n/2}(P). \end{aligned}$$

We show equation (3) by induction on n. It is obviously true for $n = 2$, because $\overline{cc} = cc$ and $\overline{d} = d$. So suppose (3) holds for all words of even weight less than n, and let v be a word of weight n (n even). Since $n - i_k = \sum_{r=0}^{k-1}(i_r + 2j_r)$, and $A^v = 0$ unless each i_r, $0 \leq r \leq k-1$, is even, $A^v \neq 0$ implies i_k is even also. So assume $v = c^{i_0}d^{j_0}c^{i_1}\cdots d^{j_{k-1}}c^{i_k}$, with all i_r even ($0 \leq r \leq k$).

If $i_k = 0$ write $v = wd$. By equation (2) $a_{n/2,v} - a_{(n-2)/2,v} = a_{(n-2)/2,w} - a_{(n-4)/2,w}$, which, by the induction assumption, equals $a_{(n-2)/2,\bar{w}} - a_{(n-4)/2,\bar{w}}$. By Theorem 9 for any cd word z, $A^{dz} = [0\, A^z\, 0]^T$, so

$$\begin{aligned} a_{n/2,v} - a_{(n-2)/2,v} = a_{(n-2)/2,\bar{w}} - a_{(n-4)/2,\bar{w}} &= a_{n/2,d\bar{w}} - a_{(n-2)/2,d\bar{w}} \\ &= a_{n/2,\bar{v}} - a_{(n-2)/2,\bar{v}}. \end{aligned}$$

On the other hand, if $i_k > 0$ then $a_{n/2,v} - a_{(n-2)/2,v} = \prod_{r=0}^{k} p_{i_r} = a_{n/2,\bar{v}} - a_{(n-2)/2,\bar{v}}$. So equation (3) holds for all words of even weight. □

The cd index of a polytope is a good way of representing the numerical combinatorial data of the polytope. There is a simple relation between the cd indices of dual polytopes, as there is between the flag vectors. The compelling advantage of the cd index over the flag vector is that it is of the correct dimension. The cd words precisely parametrize the generalized Dehn-Sommerville space. Furthermore this index can be computed via a shelling of the polytope. However, the shelling computation of the cd index involves subtraction. To show that the index has nonnegative coefficients we would like to interpret the individual coefficients as measuring the dimension of some vector space, or counting some geometric objects.

Finally, we mention the connection between the cd index and Kalai's parameters for polytopes. Kalai [11] defines for a polytope P a set of parameters that can be obtained from the Betti numbers of polytopes whose face lattices occur as intervals in the face lattice of P. These parameters are again linear functions of the flag vector of the polytope, and hence can be computed from the cd index and vice versa. Theorem 9 provides a direct way of computing a small subset of Kalai's parameters (those using only the entire face lattice of P) without first computing the flag vector. By Corollary 3 we can extract from the cd index of a polytope P the sum of the cd indices of its facets; applying Theorem 9 to this sum enables us to compute more of Kalai's parameters. We hope that further study of the *cd* index and of Kalai's parameters will produce a better understanding of the combinatorial structure of polytopes.

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