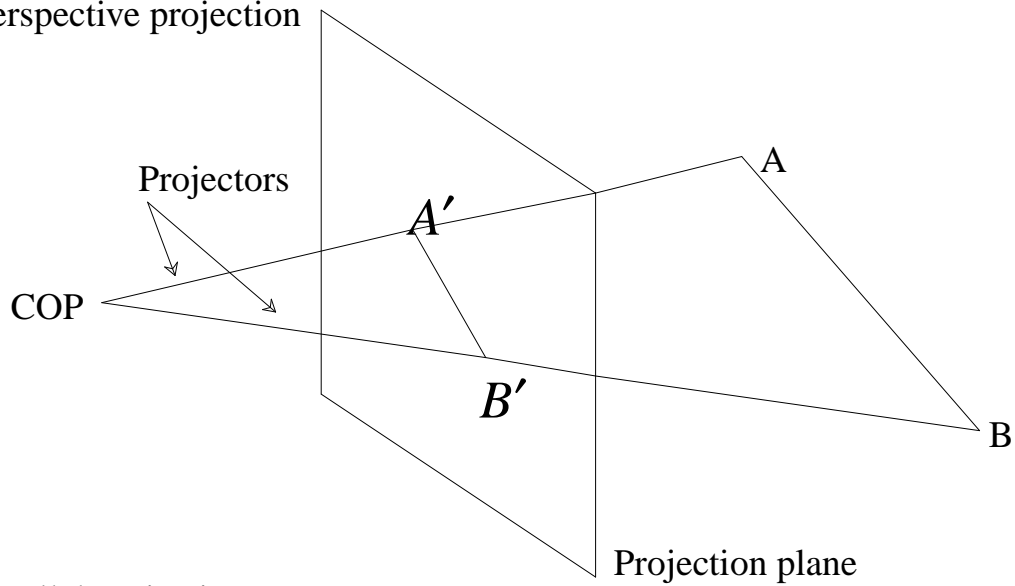


7. 3D Viewing

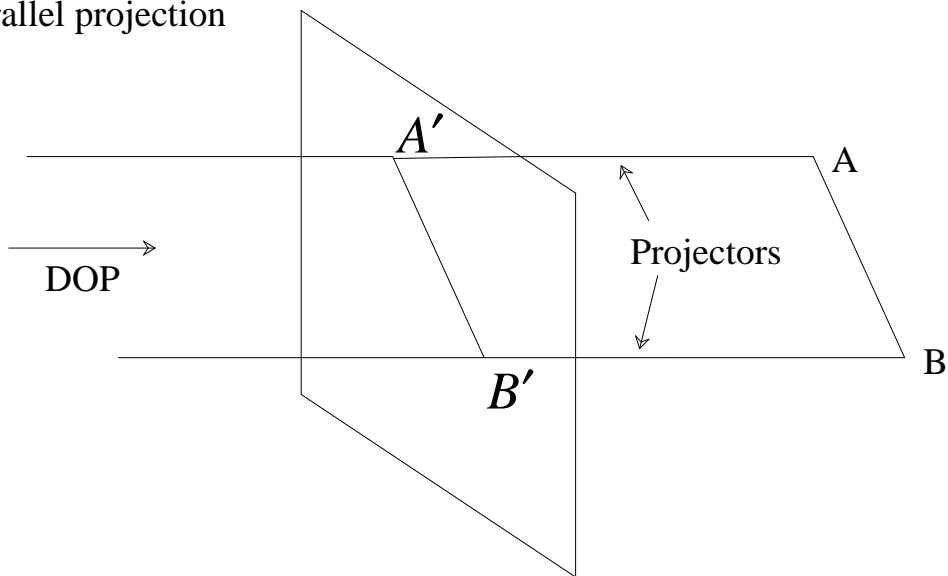
7.1 Projections:

- Why is projection necessary?

(a) Perspective projection



(b) Parallel projection

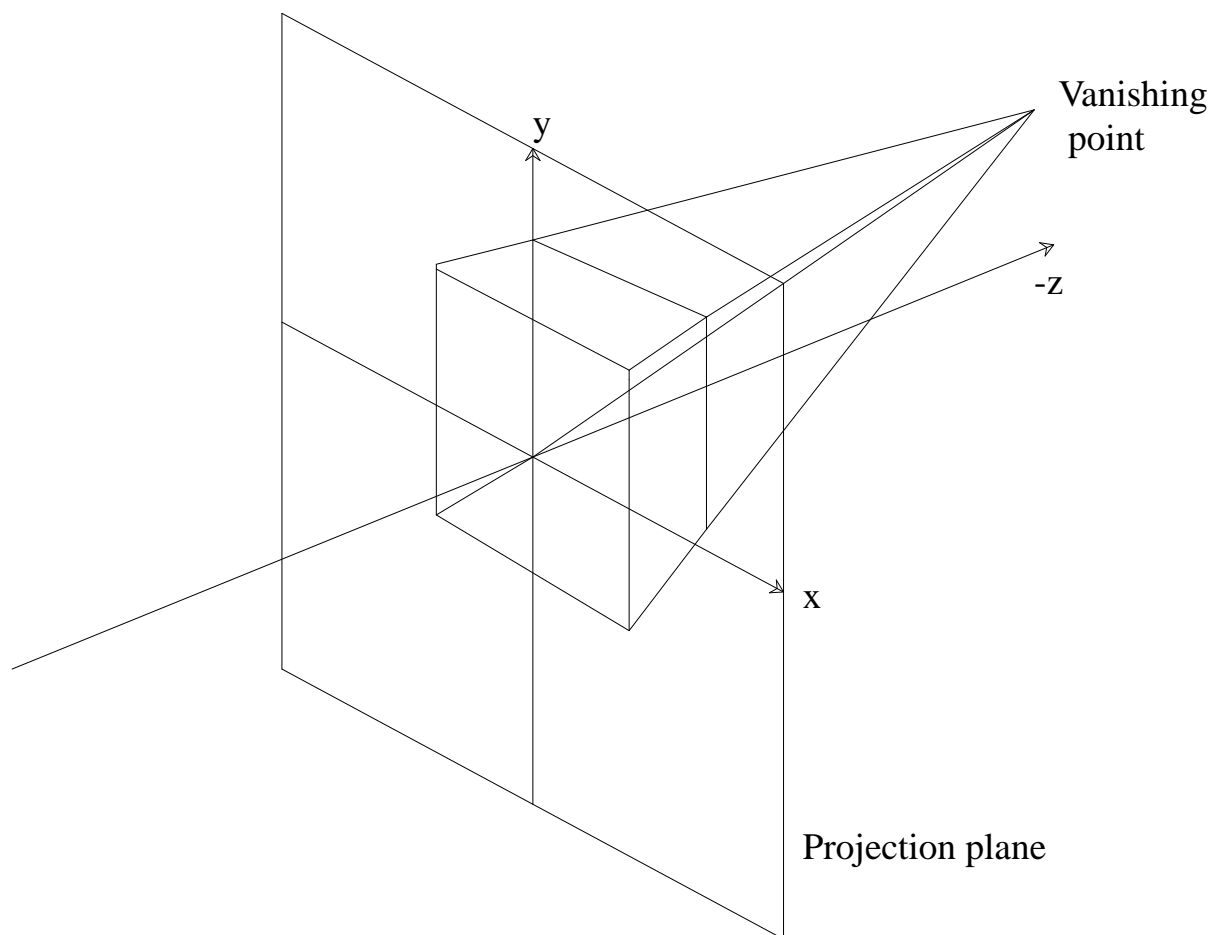


Vanishing point

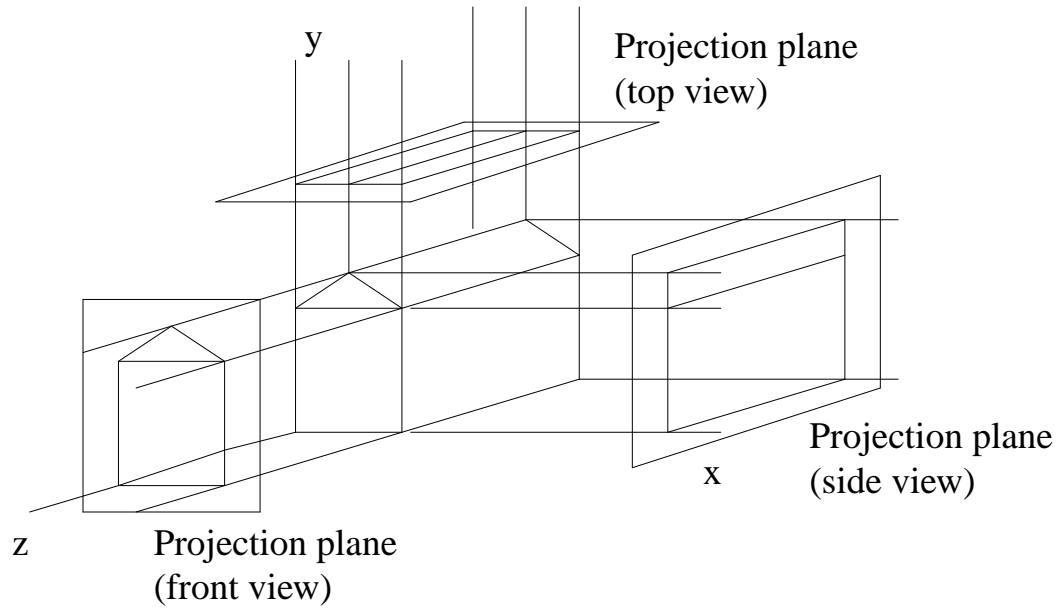
- Perspective projections of parallel lines not parallel to the projection plane will converge to a vanishing point

Principal vanishing point

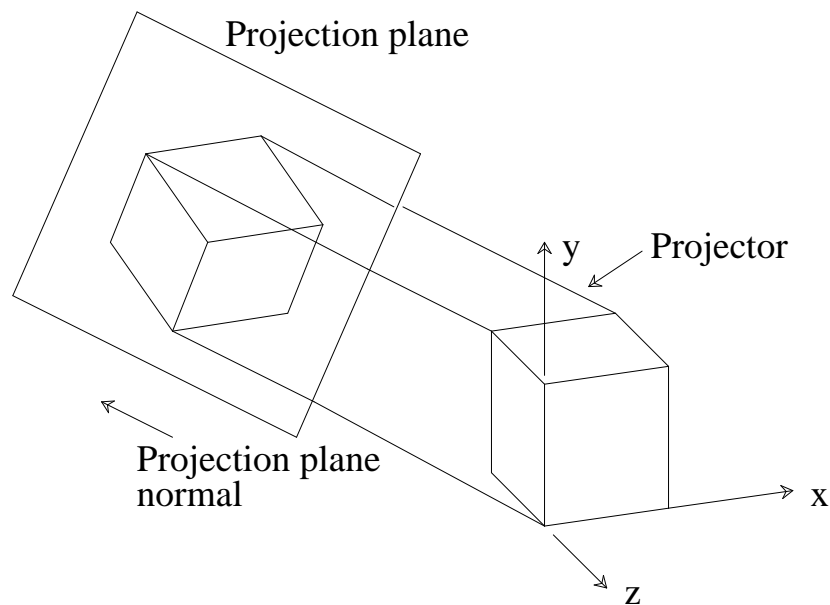
- vanishing point of a set of parallel lines that is parallel to one of the three principal axes



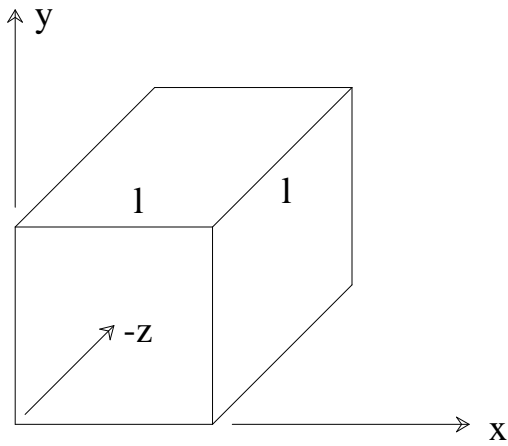
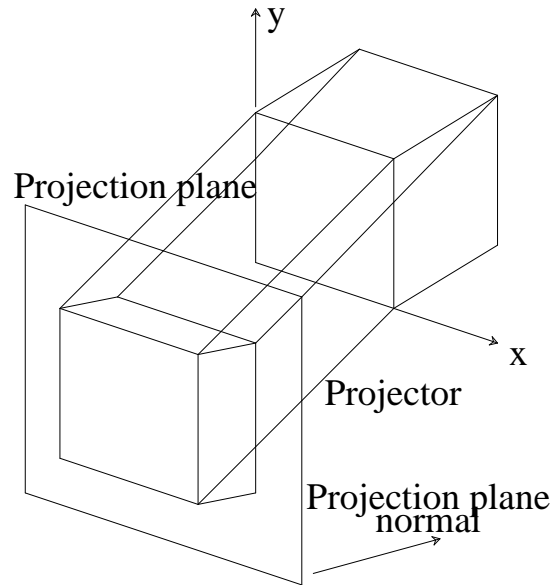
e.g., three orthographic projections



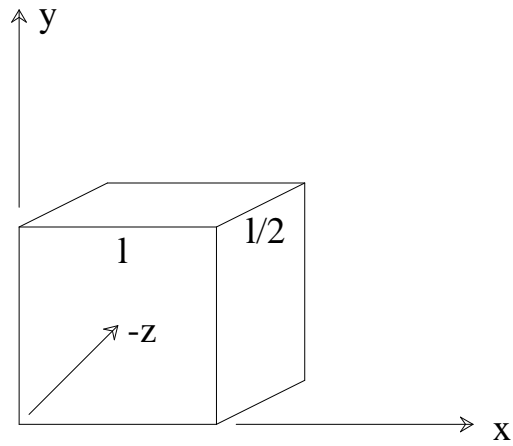
Isometric projection of a unit cube along the direction $(1, -1, -1)$.



e.g., oblique projection

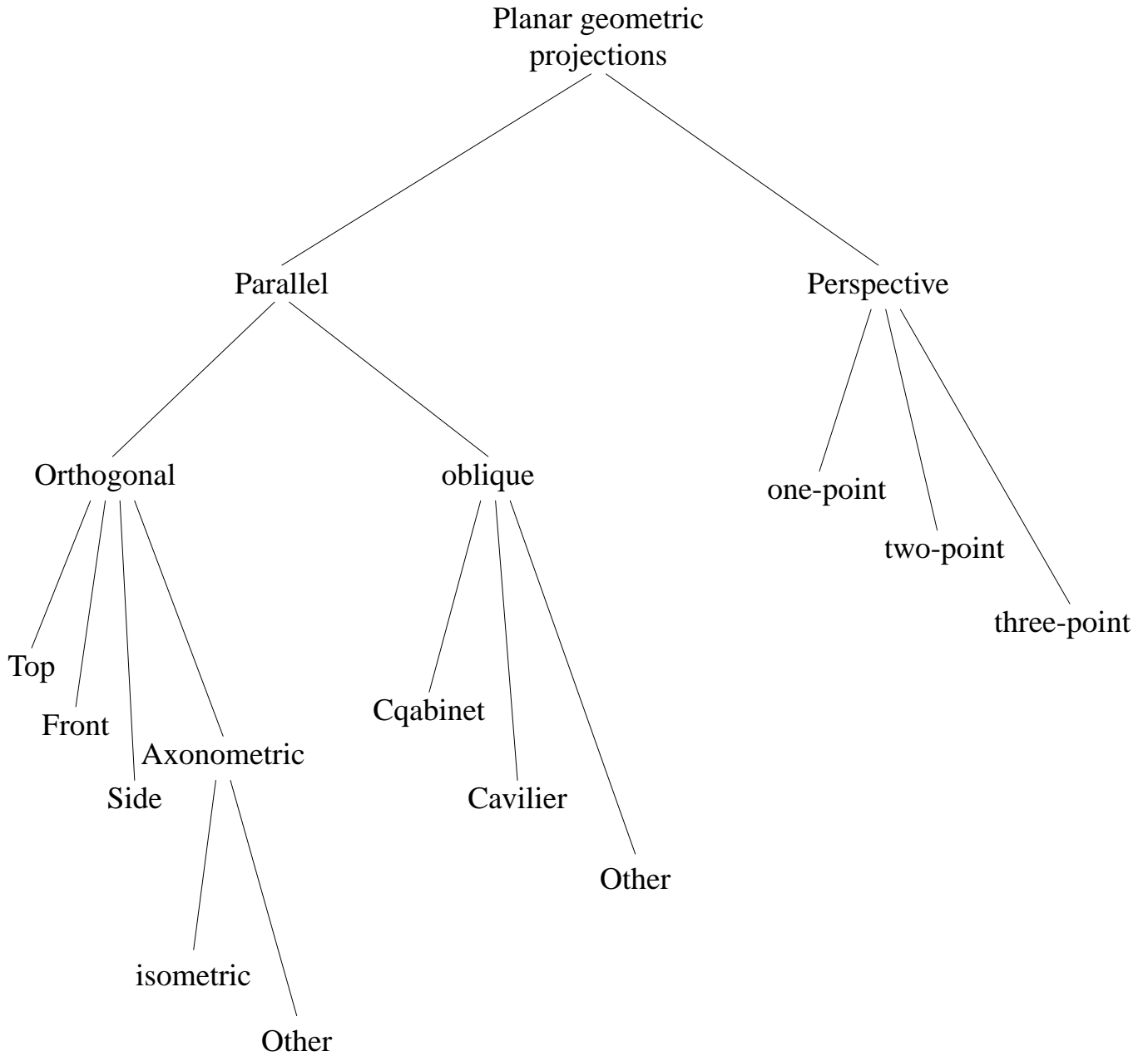


Cavalier projection



Cabinet projection

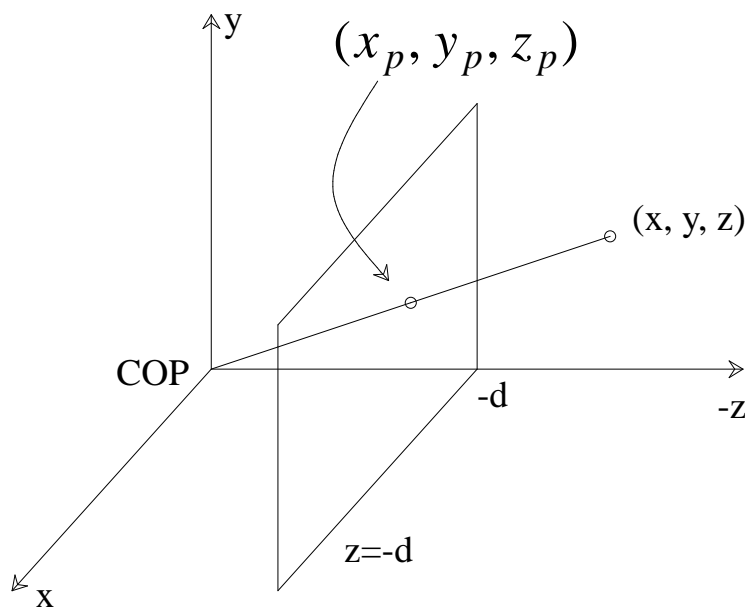
Various planar projections:



7.2 Mathematics of Projections

- Projections can be defined by 4×4 matrices
- Projection plane is normal to the z axis

Perspective projection: (not Affine, irreversible)



$$x_p = \frac{x}{-z/d}$$

$$y_p = \frac{y}{-z/d}$$

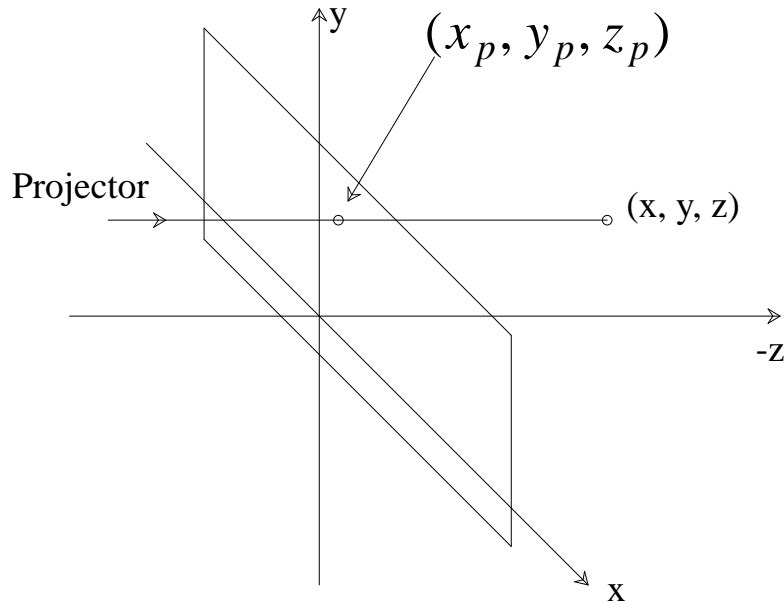
$$z_p = -d$$

$$[x_p, y_p, z_p, 1]^t = \left[\frac{x}{-z/d}, \frac{y}{-z/d}, -d, 1 \right]^t$$

$$= [x, y, z, -z/d]^t$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1/d & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = M_{per} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

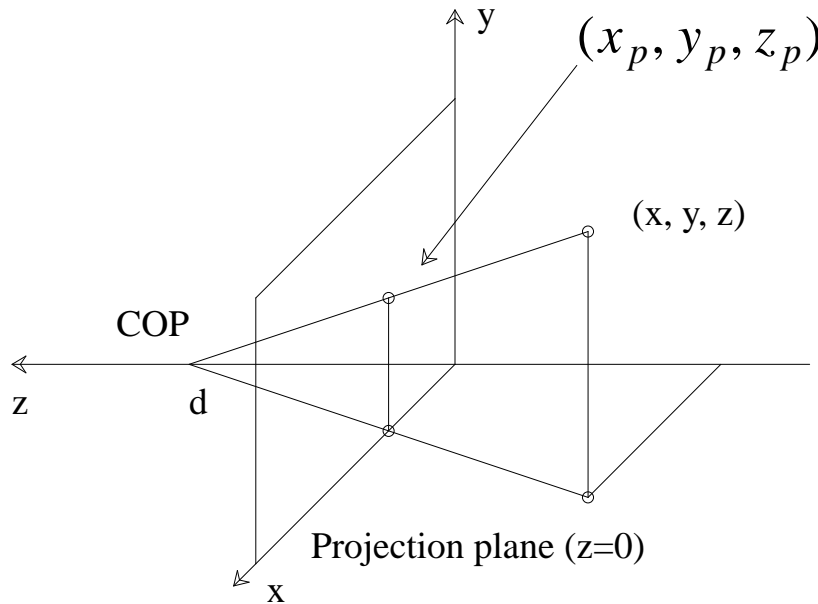
Parallel (orthogonal) projection:



$$[x_p, y_p, z_p, 1]^t = [x, y, 0, 1]^t$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = M_{par} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Perspective projection: (COP is not at the origin)



$$x_p = \frac{x}{1 - z/d}$$

$$y_p = \frac{y}{1 - z/d}$$

$$z_p = 0$$

$$[x_p, y_p, z_p, 1]^t = \left[\frac{x}{1 - z/d}, \frac{y}{1 - z/d}, 0, 1 \right]^t$$

$$= [0, 0, d, 1]^t + \left[\frac{x}{1 - z/d}, \frac{y}{1 - z/d}, -d, 1 \right]^t$$

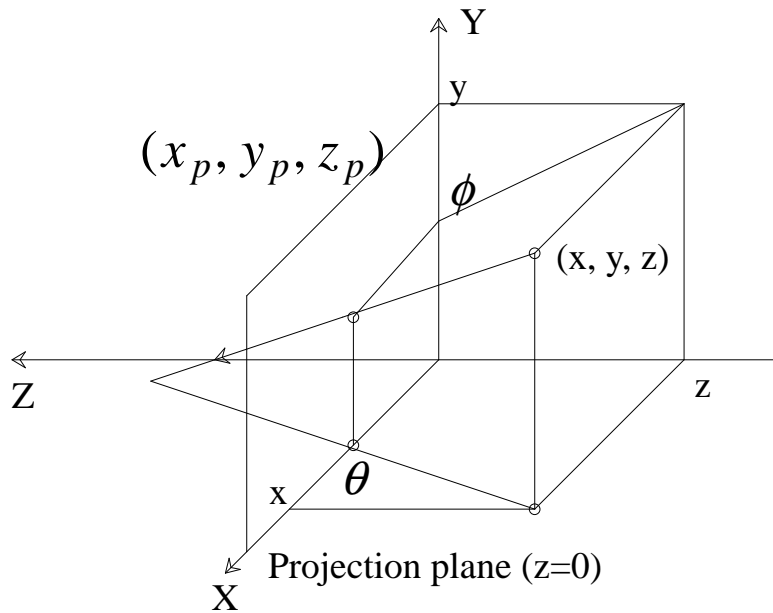
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1/d & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z - d \\ 1 \end{bmatrix}$$

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$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1/d & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -d \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$= M_t M_{per} M_t \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Oblique projection:



$$x_p = x - z \cdot \cot \theta$$

$$y_p = y - z \cdot \cot \phi$$

$$[x_p, y_p, z_p, 1]^t = [x - z \cdot \cot\theta, y - z \cdot \cot\phi, 0, 1]^t$$

$$= \begin{bmatrix} 1 & 0 & -\cot\theta & 0 \\ 0 & 1 & -\cot\phi & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -\cot\theta & 0 \\ 0 & 1 & -\cot\phi & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$= M_{par} \cdot M_{shearing} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Where are the vanishing points?

- Parallel lines after perspective projection are still parallel lines if they are also parallel to the projection plane

Why?

If $L(t) = A + \mathbf{c} \cdot t$ is a line

$$A = (A_x, A_y, A_z) ; \quad \mathbf{c} = (c_x, c_y, c_z)$$

if the view point (eye) is at the origin and the projection plane is perpendicular to the z axis at $-d$, then the perspective projection of $L(t)$ is:

$$L_p(t) = \left(d \frac{A_x + c_x t}{-A_z - c_z t}, \quad d \frac{A_y + c_y t}{-A_z - c_z t} \right) \quad (*)$$

If $L(t)$ is parallel to the projection plane ($c_z = 0$) then

$$L_p(t) = \frac{-d}{A_z} (A_x + c_x t, \quad A_y + c_y t)$$

Slope of $L_p(t)$ is c_y / c_x .

- Parallel lines after perspective projection are no longer parallel lines if they are not parallel to the projection plane

Why?

If $L(t)$ is not parallel to the projection plane ($c_z \neq 0$) then from (*) we have that

$$L_p(t) \rightarrow -d \left(\frac{c_x}{c_z}, \frac{c_y}{c_z} \right) \quad \text{when } t \rightarrow \infty$$

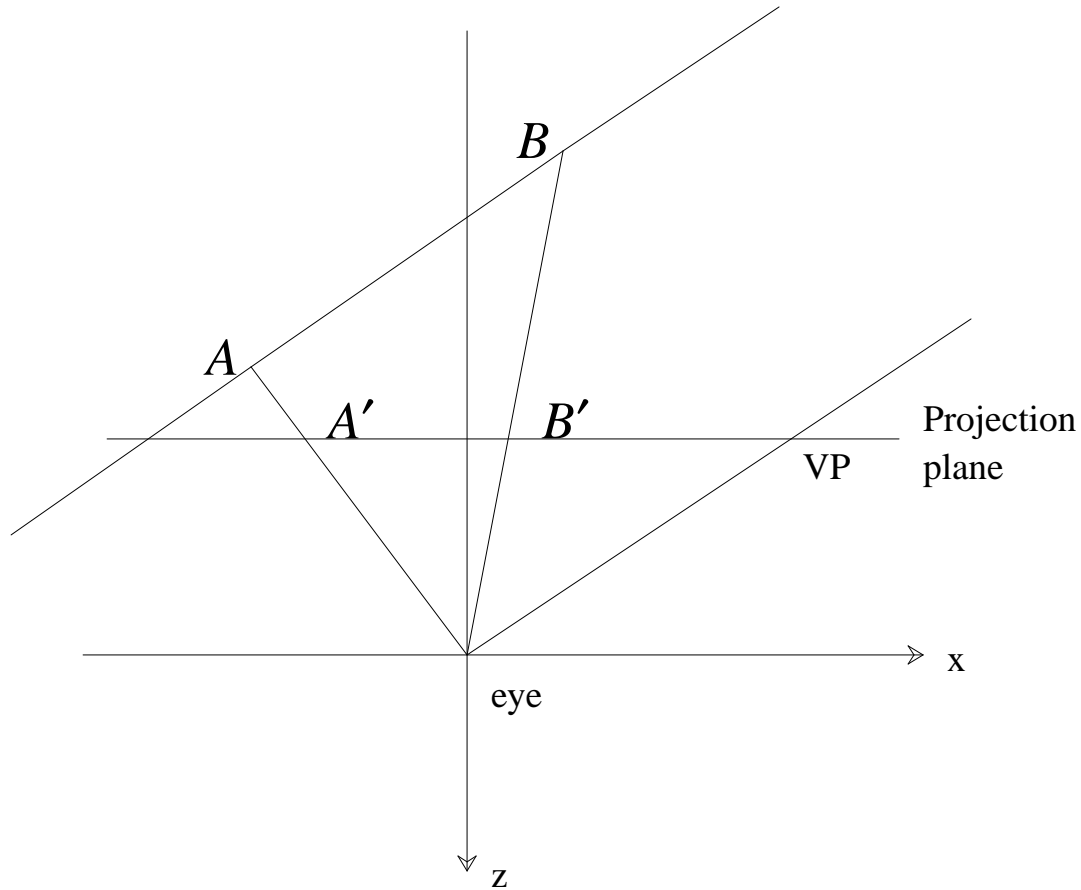
Hence, any line with the same direction vector would converge to this (vanishing) point

$$-d \left(\frac{c_x}{c_z}, \frac{c_y}{c_z} \right).$$

Principal vanishing point : vanishing point generated by lines parallel to one of the principal axes (at most three PVPs).

Two-point perspective projection is popular

How to find vanishing points?



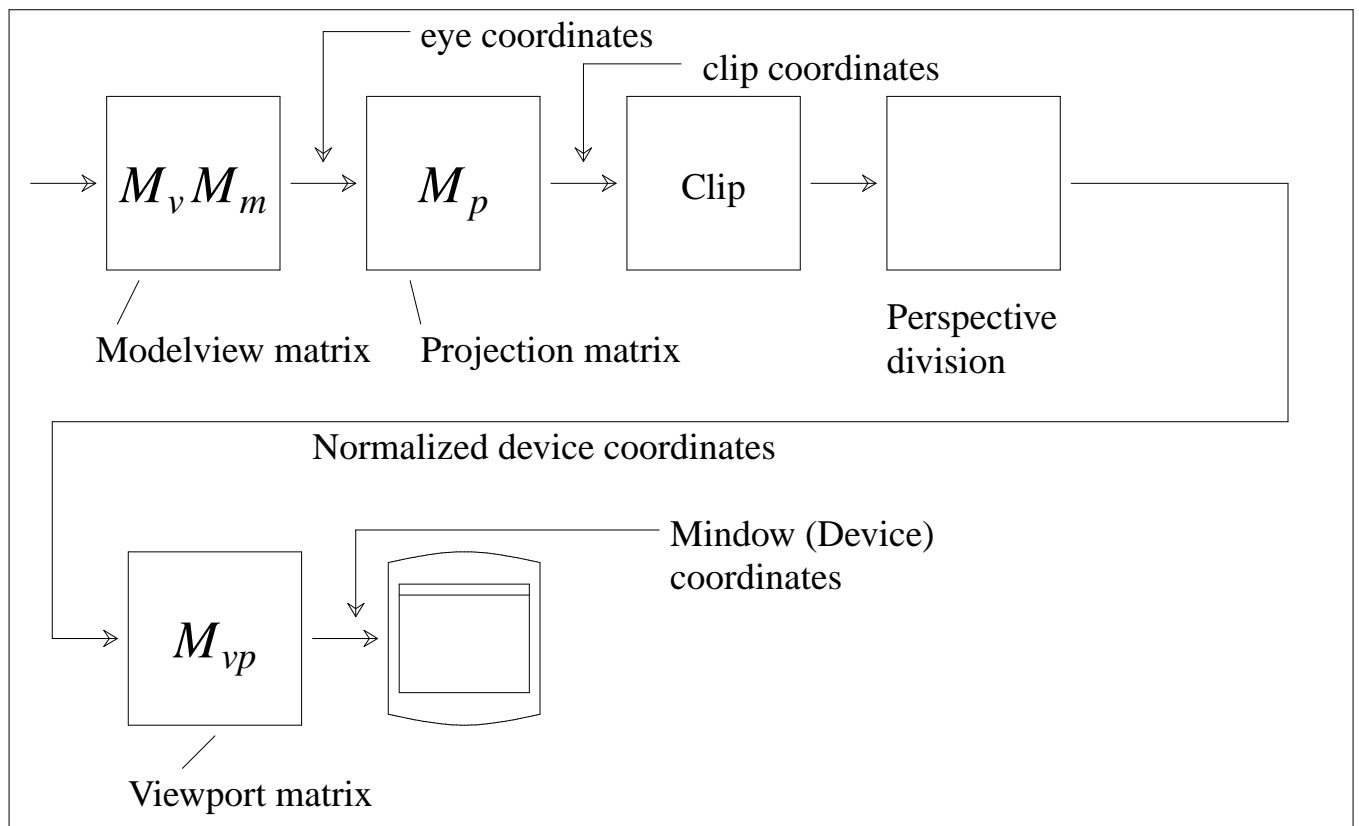
$$B' \dashrightarrow VP \quad \text{when} \quad B \dashrightarrow \infty$$

Construct a line parallel to AB that passes thru the view point (eye). The intersection of this line with the projection plane is the vanishing point of AB .

7.3 Camera Model for Perspective View

- How to create a perspective view of a scene in OpenGL?
- How to control the camera's position and orientation in OpenGL?

Conceptual model of 3D viewing:



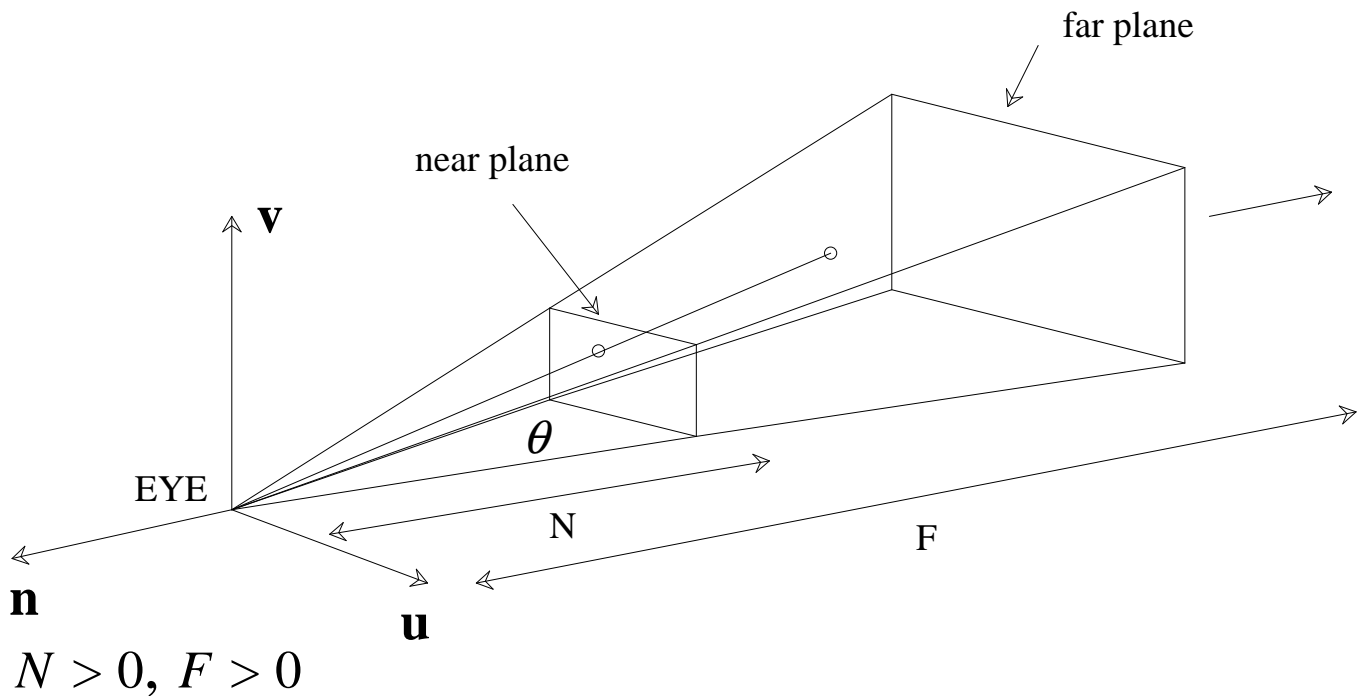
Define the view volume:

(create a camera model)

```
glMatrixMode ( GL_PROJECTION );
```

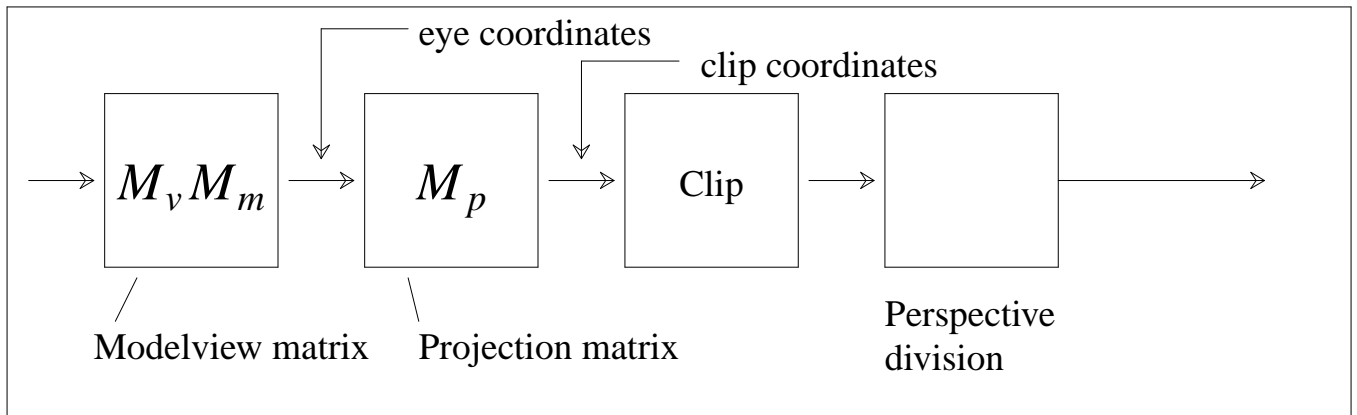
```
glLoadIdentity ( );
```

```
gluPerspective ( viewAngle, aspectRatio, N, F );
```



7.4 Building Viewing Matrix

View Pipeline



Canonical View Volume

- Parallel: $x = \pm 1$, $y = \pm 1$, $z = \pm 1$
- Perspective: $x = z$, $x = -z$, $y = z$, $y = -z$,
 $z = -z_{\min}$, $z = -1$

Modelview Matrix ($M_v M_m$):

Modeling part (M_m):

- embodies all the modeling transformations for the object

Viewing part (M_v):

- accounts for the WC to VC transformation set by the camera's position and orientation

$$M_v = \begin{bmatrix} u_x & u_y & u_z & d_x \\ v_x & v_y & v_z & d_y \\ n_x & n_y & n_z & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where

$$(d_x, d_y, d_z) = (-\mathbf{u} \cdot \mathbf{eye}, -\mathbf{v} \cdot \mathbf{eye}, -\mathbf{n} \cdot \mathbf{eye})$$

Projection Matrix(M_p):

$$M_p = \textit{scaling2}$$

* *translation*

* *perspective transformation*

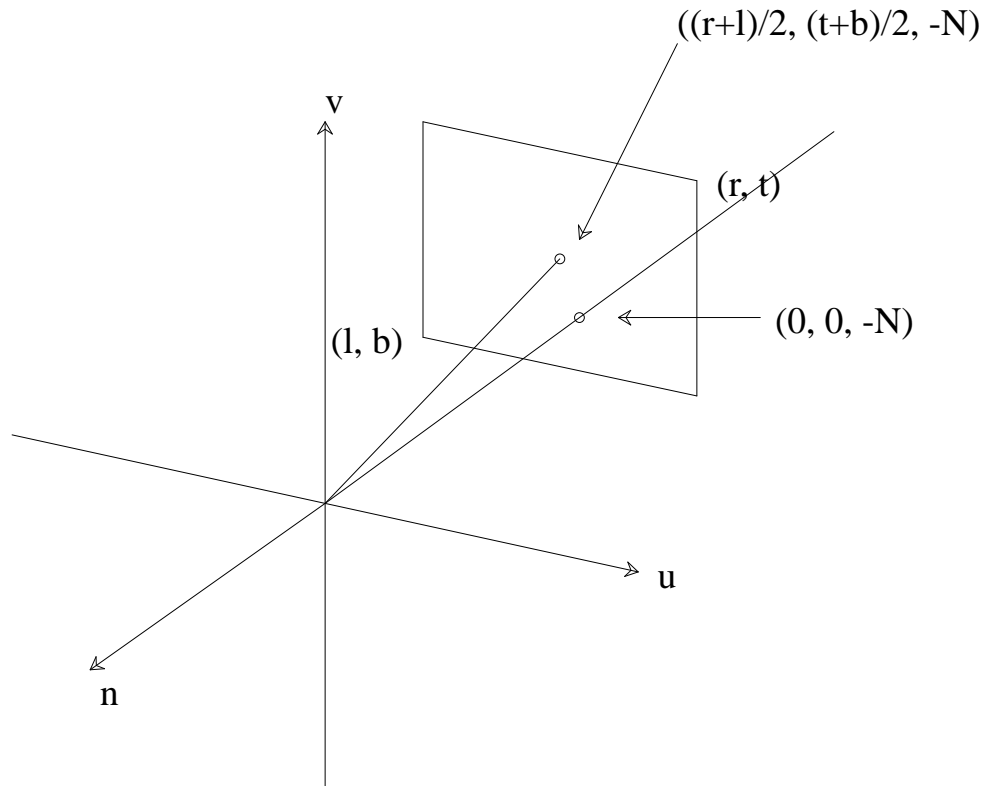
* *scaling1*

* *shearing*

$$M_p = M_{s2} * M_t * M_{pt} * M_{s1} * M_{sh}$$

Shearing:

- shear so that the center of the window would coincide with $(0, 0, -N)$



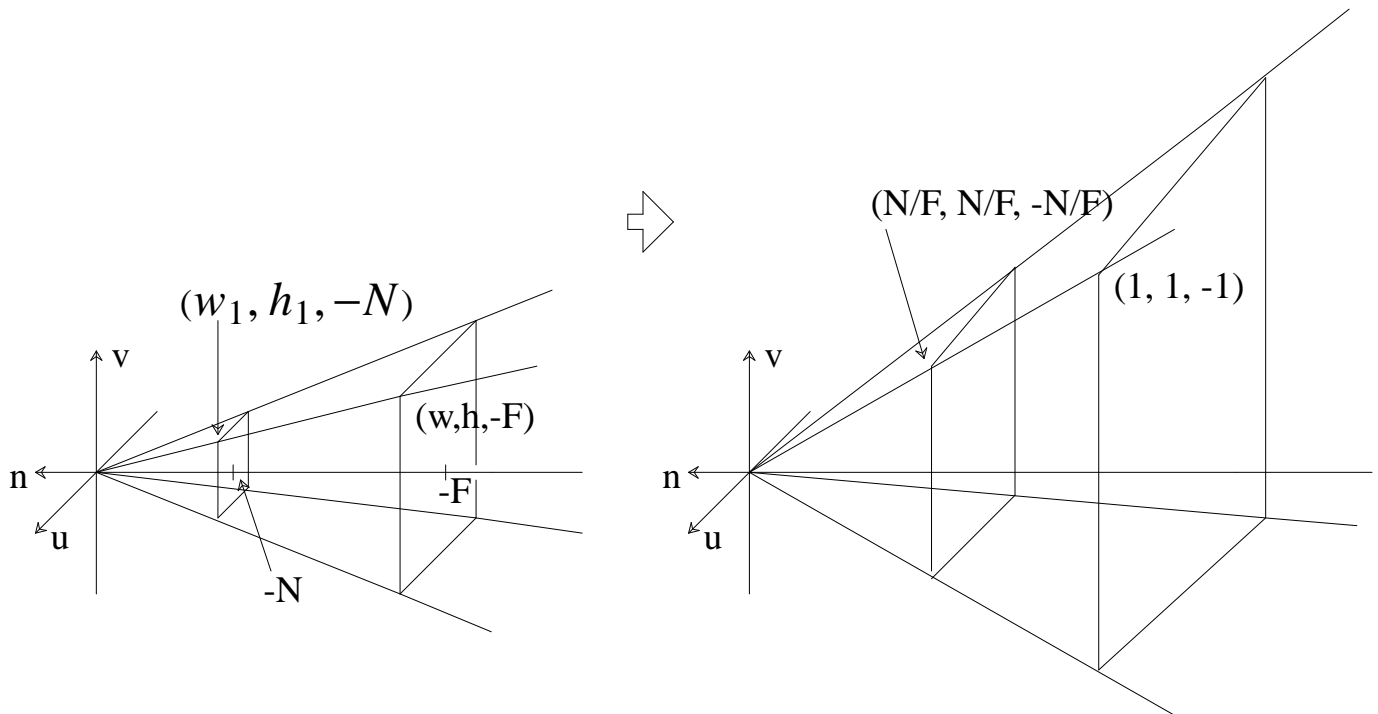
$$M_{sh} = \begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$a = \frac{r+l}{2N} = 0$$

$$b = \frac{t+b}{2N} = 0$$

Scaling1:

- scale so the user defined truncated view volume would coincide with the canonical view volume for perspective projection



$$M_{s1} = \begin{bmatrix} 1/w & 0 & 0 & 0 \\ 0 & 1/h & 0 & 0 \\ 0 & 0 & 1/F & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

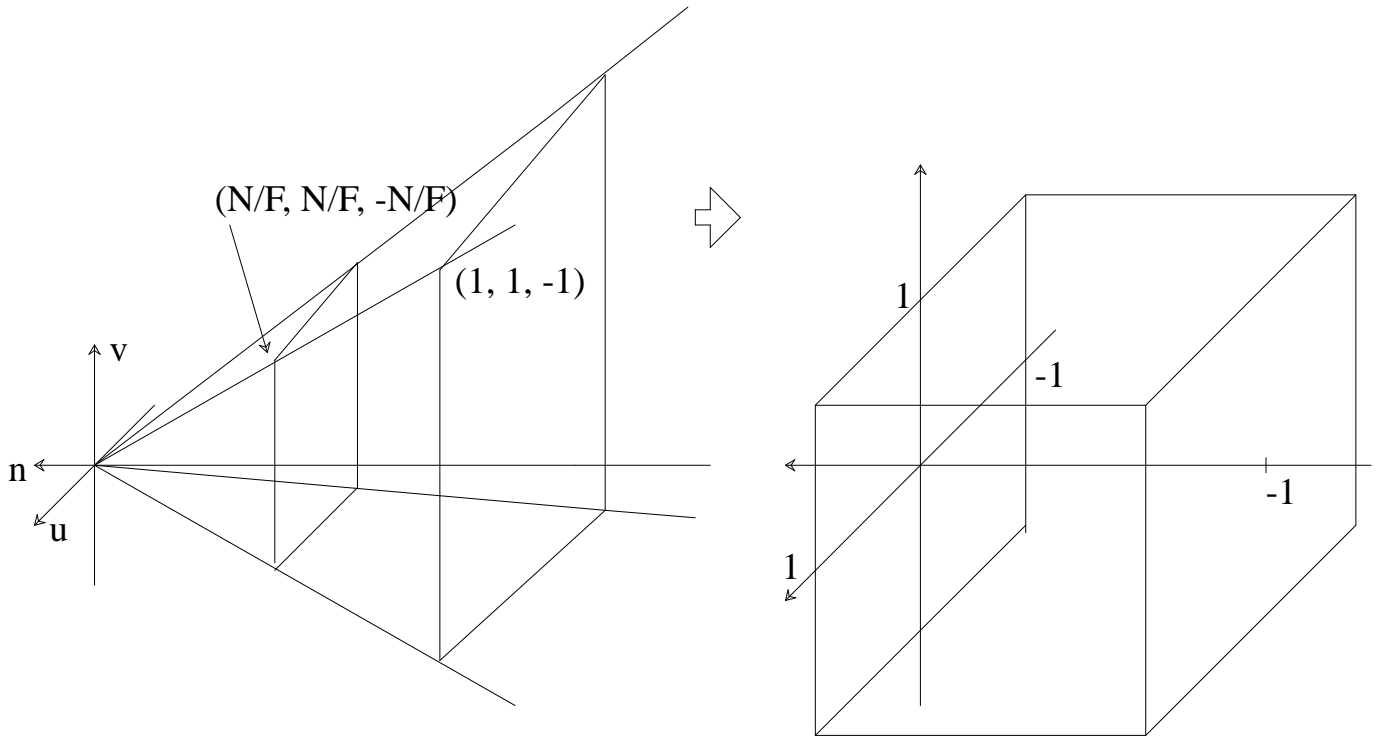
$$w = F \tan(\theta/2) AR$$

$$h = F \tan(\theta/2)$$

$$AR = \textit{aspect ratio}$$

Perspective Transformation:

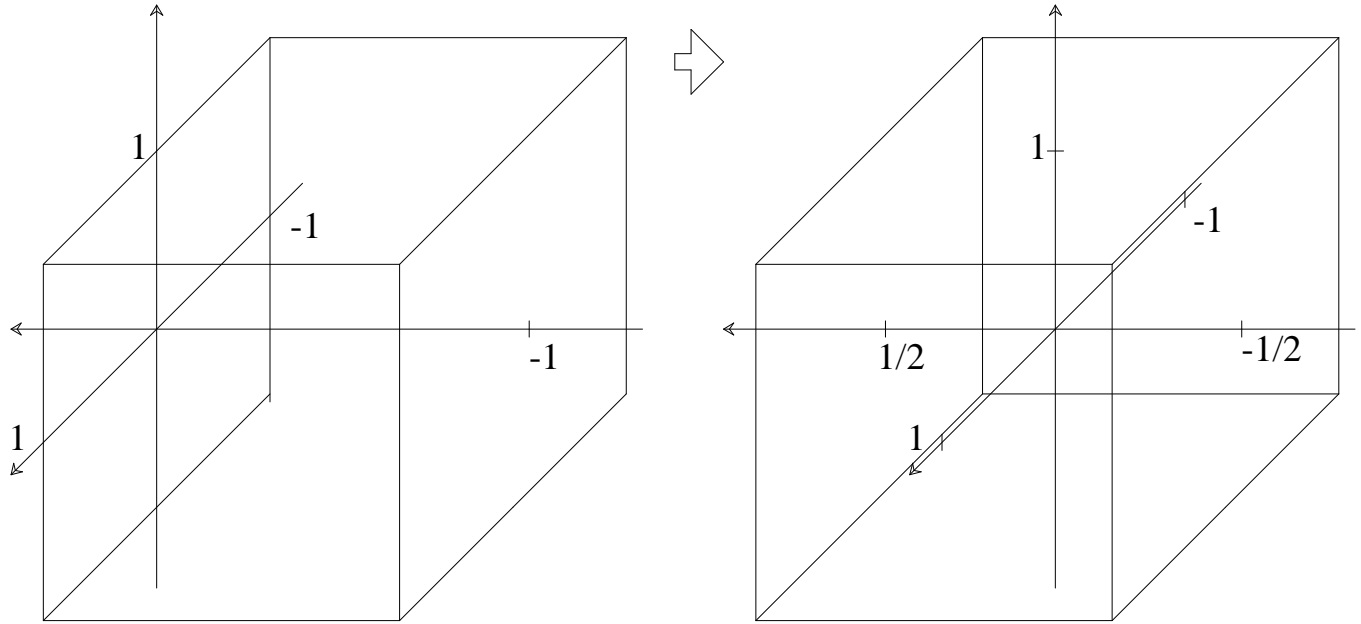
- convert CVV for perspective projection to a quasi-CVV for parallel projection



$$M_{pt} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{F}{F-N} & \frac{N}{F-N} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Translation:

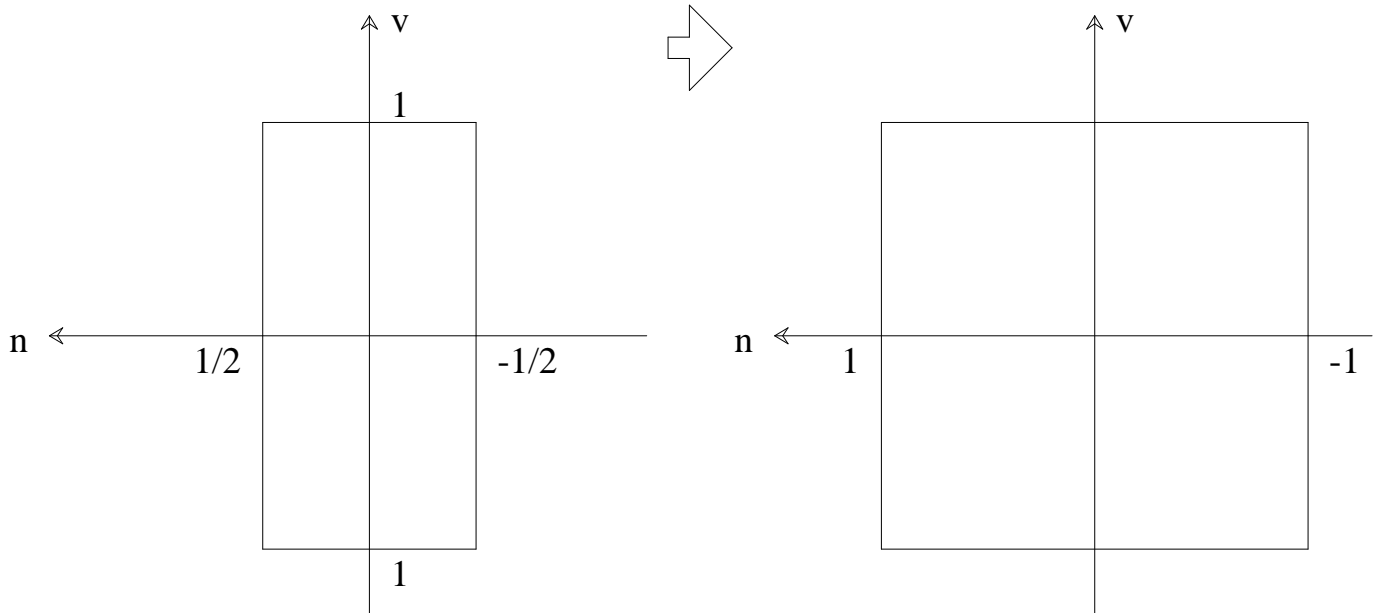
- translate center of the quasi-CVV to the origin (0,0,0)



$$M_t = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Scaling2:

- scale z-direction by 2 to get the CVV for parallel projection



$$M_{s2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

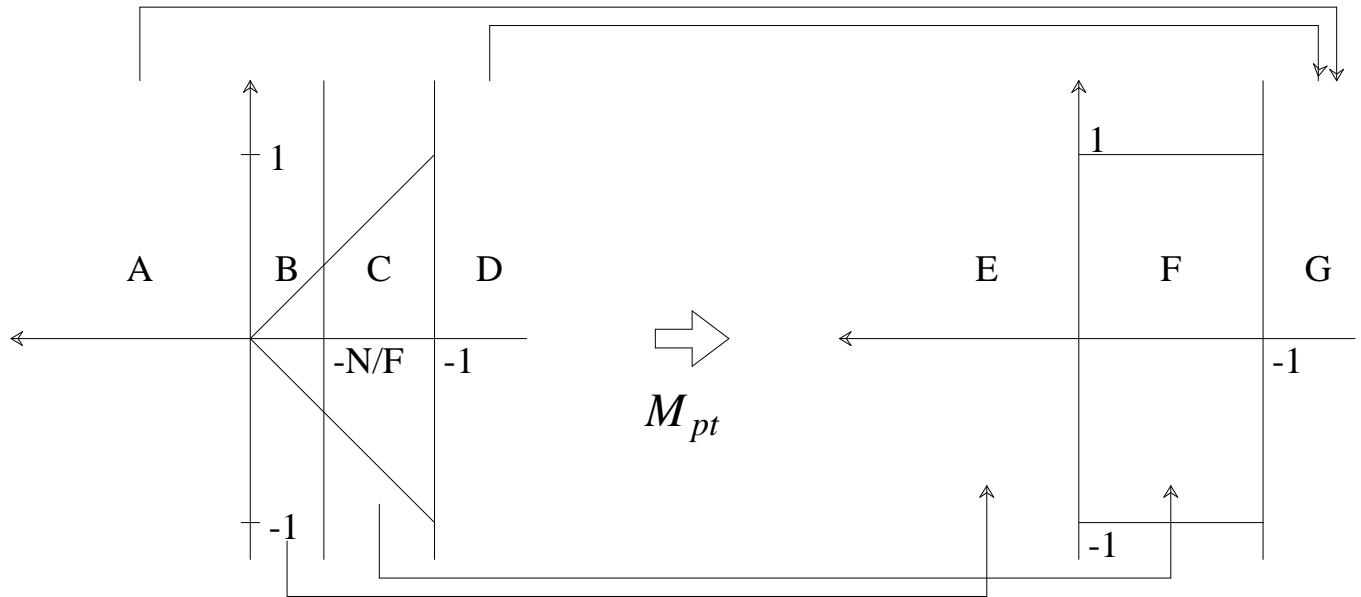
$$M_p = ?$$

$$M_p = M_{s2} * M_t * M_{pt} * M_{s1} * M_{sh}$$

$$= \frac{1}{F} \begin{bmatrix} \frac{F}{w} & 0 & 0 & 0 \\ 0 & \frac{F}{h} & 0 & 0 \\ 0 & 0 & \frac{F+N}{F-N} & \frac{2FN}{F-N} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Clipping in Homogeneous Coordinates:

- What does M_{pt} do?



$$A \text{ --- } M_{pt} \text{ --- } > G$$

$$B \text{ --- } M_{pt} \text{ --- } > E$$

$$C \text{ --- } M_{pt} \text{ --- } > F$$

$$D \text{ --- } M_{pt} \text{ --- } > G$$

Why?

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{F}{F-N} & \frac{N}{F-N} \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ \frac{Fz+N}{F-N} \\ -z \end{bmatrix} = \begin{bmatrix} -x/z \\ -y/z \\ \frac{Fz+N}{-z(F-N)} \\ 1 \end{bmatrix}$$

1.

$$\text{If } z > 0 \text{ then } \frac{Fz+N}{-z(F-N)} < -1$$

2.

$$\text{If } z < -1 \text{ then } \frac{Fz+N}{-z(F-N)} < -1$$

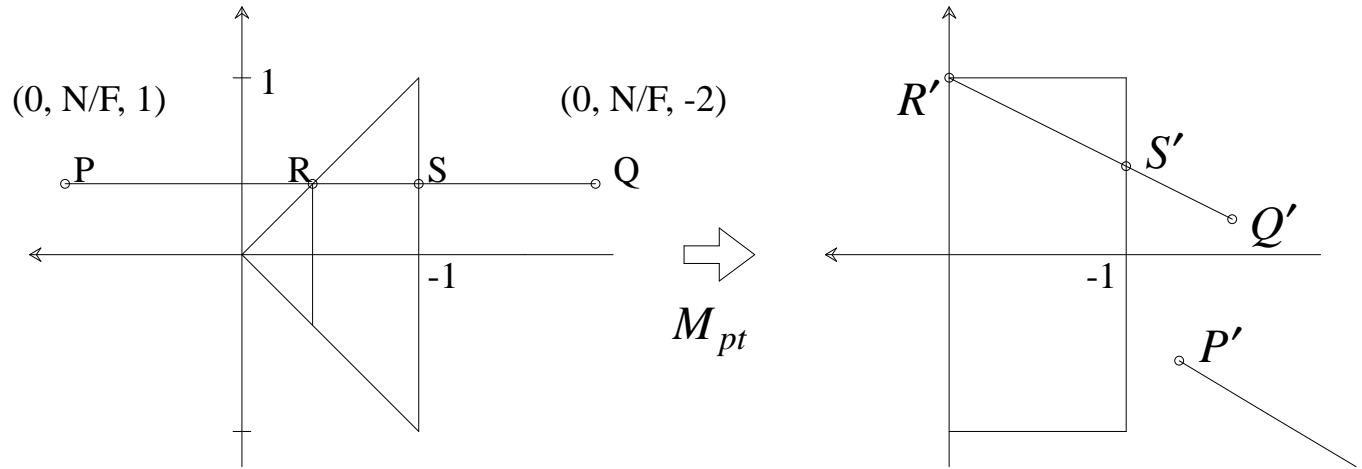
3.

$$\text{If } -\frac{N}{F} < z < 0 \text{ then } \frac{Fz+N}{-z(F-N)} > 0$$

4.

$$\text{If } -1 < z < -\frac{N}{F} \text{ then } 0 > \frac{Fz+N}{-z(F-N)} > -1$$

Now consider the following example:



M_{pt} maps P and Q both into points in region G

$$M_{pt} P = \begin{bmatrix} 0 \\ \frac{N/F}{F+N} \\ \frac{F-N}{-1} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{-N/F}{-F-N} \\ \frac{F-N}{1} \end{bmatrix}$$

$$M_{pt} Q = \begin{bmatrix} 0 \\ \frac{N/F}{-2F+N} \\ \frac{F-N}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{N/2F}{-2F+N} \\ \frac{2(F-N)}{1} \end{bmatrix}$$

- If we clip the line segment $P'Q'$ against the CVV after the **perspective division**, since P' and Q' are both to the right of the far clipping plane, we would think the entire line segment is outside the CVV and would consequently have the line discarded. But $R'S'$ of the line is actually inside the CVV.
- The reason that this happens is because the division performed for P' changes the sign of z-component from positive to negative.

Remedy: perform *clipping* before performing *perspective division*, i.e., clip in homogeneous coordinates, then perform perspective division.

How to clip a point in homogeneous coordinates?

If we use $M_{pt} \equiv M_{s2} * M_t * M_{pt}$ as the *perspective transformation*

$$M_{pt} = M_{s2} * M_t * M_{pt} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{F+N}{F-N} & \frac{2N}{F-N} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

(what is the difference between M_{pt} and M_{pt} ?)

then a point

$$(X, Y, Z, W)^t = M_{pt}(x, y, z, 1)^t$$

(before the **perspective division**) is inside the CVV for parallel projection if

$$-1 \leq \frac{X}{W} \leq 1, \quad -1 \leq \frac{Y}{W} \leq 1, \quad -1 \leq \frac{Z}{W} \leq 1$$

If $W > 0$, this means the **boundary coordinates** (BC's) must all be positive:

Boundary coordinate	homogeneous value	clip plane
BC_0	$W + X > 0$	$x = -1$
BC_1	$W - X > 0$	$x = 1$
BC_2	$W + Y > 0$	$y = -1$
BC_3	$W - Y > 0$	$y = 1$
BC_4	$W + Z > 0$	$z = -1$
BC_5	$W - Z > 0$	$z = 1$

If $W < 0$, then all the BC's must be negative:

Boundary coordinate	homogeneous value	clip plane
BC_0	$W + X < 0$	$x = -1$
BC_1	$W - X < 0$	$x = 1$
BC_2	$W + Y < 0$	$y = -1$
BC_3	$W - Y < 0$	$y = 1$
BC_4	$W + Z < 0$	$z = -1$
BC_5	$W - Z < 0$	$z = 1$

How to clip a line segment in homogeneous coordinates?

Use 3D Cyrus-Beck clipper:

input: $A = (A_x, A_y, A_z, A_w)$

$$B = (B_x, B_y, B_z, B_w)$$

$$L(t) = A + (B - A) * t$$

- Compute BC 's for A and B
- Compute *outcodes* for A and B
- Perform "trivial rejection" test
- Perform "trivial acceptance" test

If both tests fail, then

- for each clip plane, if A and B are on different side of the plane, find parameter of the *intersection point (hit point)*. For instance, for $x = 1$ the hit point's parameter is

$$t = \frac{A_w - A_x}{(A_w - A_x) - (B_w - B_x)}$$

- Then update related items' values