1. Solve the following system of linear equations using Gaussian elimination without pivoting, in the floating-point number system \( F(10,5,-128,127) \) with chopped arithmetic.

\[
\begin{align*}
2x_1 - 3x_2 + 4x_3 &= 8 \quad \text{......... (1)} \\
5x_1 - 7.501x_2 + x_3 &= -7.002 \quad \text{......... (2)} \\
4x_1 + x_2 + 2x_3 &= 12 \quad \text{......... (3)}
\end{align*}
\]

Sol. First, eliminate entries in the first column that are below the diagonal line:

\[
\begin{align*}
(2) - (1)*\frac{5}{2} \\
5x_1 - 7.501x_2 + x_3 &= -7.002 \\
-\left(5x_1 - 7.5x_2 + 10x_3 = 20\right) \\
\hline
-0.001x_2 - 9x_3 &= -27.002
\end{align*}
\]

\[
(3) - (1)*\frac{4}{2} \\
4x_1 + x_2 + 2x_3 &= 12 \\
-\left(4x_1 - 6x_2 + 8x_3 = 16\right) \\
\hline
7x_2 - 6x_3 &= -4
\]

Hence, we have

\[
\begin{align*}
2x_1 - 3x_2 + 4x_3 &= 8 \quad \text{......... (1)} \\
- 0.001x_2 - 9x_3 &= -27.002 \quad \text{......... (2)} \\
7x_2 - 6x_3 &= -4 \quad \text{......... (3)}
\end{align*}
\]

Then eliminate the entry in the second column that is below the main diagonal:
2. Solve the above system of linear equations using Gaussian elimination with pivoting, in the floating-point number system $F(10,5,-128,127)$ with chopped arithmetic.

Sol. First, switch Eq.(1) and Eq.(2) to make Eq.(2) the pivot equation of column one:

$$
\begin{align*}
5x_1 - 7.501x_2 + x_3 &= -7.000 \\
2x_1 - 3x_2 + 4x_3 &= 8 \\
4x_1 + x_2 + 2x_3 &= 12
\end{align*}
$$

Next, eliminate entries in the first column that are below the diagonal line:
(2) - (1)*\[\frac{2}{5}\]
\[
\begin{align*}
2x_1 - 3.0 & \quad x_2 + 4.0x_3 = 8.0 \\
-(2x_1 - 3.0004x_2 - 0.4x_3) & = -2.8008
\end{align*}
\]
\[
0.0004x_2 + 3.6x_3 = 10.800
\]

(3) - (1)*\[\frac{4}{5}\]
\[
\begin{align*}
4x_1 + x_2 + 2.0x_3 = 12.0 \\
-(4x_1 - 6.0008x_2 + 0.8x_3) & = -5.6016
\end{align*}
\]
\[
7.0008x_2 + 1.2x_3 = 17.601
\]

We get
\[
\begin{align*}
5x_1 - 7.501x_2 + x_3 & = -7.002 \quad \text{......... (1)} \\
0.0004x_2 + 3.6x_3 & = 10.800 \quad \text{......... (2)} \\
7.0008x_2 + 1.2x_3 & = 17.601 \quad \text{......... (3)}
\end{align*}
\]

We need to switch eq. (2) with eq. (3), to make eq. (3) the pivot equation for stage 2.
\[
\begin{align*}
5x_1 - 7.501x_2 + x_3 & = -7.002 \quad \text{......... (1)} \\
7.0008x_2 + 1.2x_3 & = 17.601 \quad \text{......... (2)} \\
0.0004x_2 + 3.6x_3 & = 10.800 \quad \text{......... (3)}
\end{align*}
\]

Then, perform
\[
(3) - (2)*\frac{0.0004}{7.0008} = 0.000057163
\]
\[
\begin{align*}
-0.0004x_2 + 3.6000 & \quad x_3 = 10.800 \\
-(0.0004x_2 + 0.000068563x_3 = 0.0010056) &
\end{align*}
\]
\[
3.5999x_3 = 10.798
\]

Hence, the upper triangular system is:
\[
\begin{align*}
5x_1 - 7.501x_2 + x_3 & = -7.002 \quad \text{......... (1)} \\
7.0008x_2 + 1.2x_3 & = 17.601 \quad \text{......... (2)} \\
3.5999x_3 & = 10.798 \quad \text{......... (3)}
\end{align*}
\]
Back Substitution:

\[
\begin{align*}
x_3 &= \frac{10.798}{3.5999} = 2.9995 \\
x_2 &= \frac{17.601 - 1.2x_3}{7.0008} = \frac{17.601 - 1.2 \times 2.9995}{7.0008} = \frac{14.001}{7.0008} = 1.9999 \\
x_3 &= \frac{-7.0028 - x_3 + 7.501x_2}{5} = \frac{-7.002 - 2.9995 + 7.501 \times 1.9999}{5} = \frac{10.001 + 15.001}{5} \\
&= \frac{5}{5} = 1
\end{align*}
\]

3. Given \((x_0, f_0) = (1, 1), (x_1, f_1) = (2, 1)\) \((x_2, f_2) = (4, 1), (x_3, f_3) = (5, 1)\), first construct Lagrange polynomials \(l_0(x), l_1(x), l_2(x)\) and \(l_3(x)\), then construct the Lagrange form of \(p_3(x)\), the polynomial that interpolates \((x_0, f_0), (x_1, f_1), (x_2, f_2)\) and \((x_3, f_3)\). What is the value of \(p_3(x)\) at \(x = 3\)?

**Sol.** \(p_3(3) = 1\).

This follows from the fact that \(g(x) \equiv p_3(x) - 1\) is equal to zero at \(x = 1, 2, 4, 5\). Hence, we have

\[
g(x) = C (x - 1)(x - 2)(x - 4)(x - 5)
\]

for some \(C\). Since \(g(x)\) is a polynomial of degree 3, we must have \(C = 0\) (otherwise, \(g(x)\) would be a polynomial of degree 4). But then, \(p_3(x) = 1\) for all \(x\).

Note that if you compute \(p_3(x)\) directly, you would get \(p_3(x) = 1\) too.

4. Prove that for given \((x_0, f_0), (x_1, f_1), ..., (x_n, f_n)\), if the Lagrange polynomials \(l_i(x)\) are defined as follows:

\[
l_i(x) = \frac{\prod_{j=0, j \neq i}^{n} (x - x_j)}{\prod_{j=0, j \neq i}^{n} (x_i - x_j)}, \quad i = 0, 1, ..., n
\]

then

\[
\sum_{i=0}^{n} l_i(x) = 1
\]
Sol. Let \( q(x) = \sum_{i=0}^{n} l_i(x) = \sum_{i=0}^{n} 1^*l_i(x) \).

\( q(x) \) is a polynomial of degree \( n \) interpolationg \( (x_i, 1), i = 0, 1, ..., n \). Following an argument similar to that of Problem 3, we can easily prove that \( g(x) \equiv q(x) - 1 \) is a zero polynomial, i.e.,

\[
g(x) \equiv q(x) - 1 = 0, \quad \text{for all } x
\]

Therefore \( q(x) = 1 \) for all \( x \).