For instance, our second particular example would provide

k	1	2	3	4	5	6	7	8	9	10
a_k	11	8	7	1	9	6	5	10	3	12
x_k	1	2	3	4	2	4	5	2	6	1
y_k	1	1	1	1	2	2	2	3	2	4

If, in general, there is no decreasing or increasing subsequence of length $n+1$, then $1 \le x_k \le n$ and $1 \le y_k \le n$ for all $1 \le k \le n^2+1$. Consequently, there are at most n^2 *distinct* ordered pairs (x_k, y_k). But we have n^2+1 ordered pairs (x_k, y_k), since $1 \le k \le n^2+1$. So the pigeonhole principle implies that there are two identical ordered pairs (x_i, y_i), (x_j, y_j), where $i \ne j$ — say $i < j$. Now the real numbers $a_1, a_2, \ldots, a_{n^2+1}$ are distinct, so if $a_i < a_j$ then $y_i < y_j$, while if $a_j < a_i$ then $x_j > x_i$. In either case we no longer have $(x_i, y_i) = (x_j, y_j)$. This contradiction tells us that $x_k = n+1$ or $y_k = n+1$ for some $n+1 \le k \le n^2+1$; the result then follows.

For an interesting application of this result, consider n^2+1 sumo wrestlers facing forward and standing shoulder to shoulder. (Here no two wrestlers have the same weight.) We can select $n+1$ of these wrestlers to take one step forward so that, as they are scanned from left to right, their successive weights either decrease or increase.

EXERCISES 5.5

1. In Example 5.40, what plays the roles of the pigeons and of the pigeonholes?

2. Show that if eight people are in a room, at least two of them have birthdays that occur on the same day of the week.

3. An auditorium has a seating capacity of 800. How many seats must be occupied to guarantee that at least two people seated in the auditorium have the same first and last initials?

4. Let $S = \{3, 7, 11, 15, 19, \ldots, 95, 99, 103\}$. How many elements must we select from S to insure that there will be at least two whose sum is 110?

5. a) Prove that if 151 integers are selected from $\{1, 2, 3, \ldots, 300\}$, then the selection must include two integers x, y where $x \mid y$ or $y \mid x$.

   b) Write a statement that generalizes the results of part (a) and Example 5.43.

6. Prove that if we select 101 integers from the set $S = \{1, 2, 3, \ldots, 200\}$, there exist m, n in the selection where $\gcd(m, n) = 1$.

7. a) Show that if any 14 integers are selected from the set $S = \{1, 2, 3, \ldots, 25\}$, there are at least two whose sum is 26.

   b) Write a statement that generalizes the results of part (a) and Example 5.44.

8. a) If $S \subseteq \mathbf{Z}^+$ and $|S| \ge 3$, prove that there exist distinct $x, y \in S$ where $x + y$ is even.

   b) Let $S \subseteq \mathbf{Z}^+ \times \mathbf{Z}^+$. Find the minimal value of $|S|$ that guarantees the existence of distinct ordered pairs $(x_1, x_2), (y_1, y_2) \in S$ such that $x_1 + y_1$ and $x_2 + y_2$ are both even.

   c) Extending the ideas in parts (a) and (b), consider $S \subseteq \mathbf{Z}^+ \times \mathbf{Z}^+ \times \mathbf{Z}^+$. What size must $|S|$ be to guarantee the existence of distinct ordered triples $(x_1, x_2, x_3), (y_1, y_2, y_3) \in S$ where $x_1 + y_1$, $x_2 + y_2$, and $x_3 + y_3$ are all even?

   d) Generalize the results of parts (a), (b), and (c).

   e) A point $P(x, y)$ in the Cartesian plane is called a *lattice point* if $x, y \in \mathbf{Z}$. Given distinct lattice points $P_1(x_1, y_1), P_2(x_2, y_2), \ldots, P_n(x_n, y_n)$, determine the smallest value of n that guarantees the existence of $P_i(x_i, y_i)$, $P_j(x_j, y_j)$, $1 \le i < j \le n$, such that the midpoint of the line segment connecting $P_i(x_i, y_i)$ and $P_j(x_j, y_j)$ is also a lattice point.

9. a) If 11 integers are selected from $\{1, 2, 3, \ldots, 100\}$, prove that there are at least two, say x and y, such that $0 < |\sqrt{x} - \sqrt{y}| < 1$.

   b) Write a statement that generalizes the result of part (a).

10. Let triangle ABC be equilateral, with $AB = 1$. Show that if we select 10 points in the interior of this triangle, there must be at least two whose distance apart is less than $1/3$.

11. Let $ABCD$ be a square with $AB = 1$. Show that if we select five points in the interior of this square, there are at least two whose distance apart is less than $1/\sqrt{2}$.

12. Let $A \subseteq \{1, 2, 3, \ldots, 25\}$ where $|A| = 9$. For any subset B of A let s_B denote the sum of the elements in B. Prove that

there are distinct subsets C, D of A such that $|C| = |D| = 5$ and $s_C = s_D$.

13. Let S be a set of five positive integers the maximum of which is at most 9. Prove that the sums of the elements in all the nonempty subsets of S cannot all be distinct.

14. During the first six weeks of his senior year in college, Brace sends out at least one resumé each day but no more than 60 resumés in total. Show that there is a period of consecutive days during which he sends out exactly 23 resumés.

15. Let $S \subset \mathbf{Z}^+$ with $|S| = 7$. For $\emptyset \neq A \subseteq S$, let s_A denote the sum of the elements in A. If m is the maximum element in S, find the possible values of m so that there will exist distinct subsets B, C of S with $s_B = s_C$.

16. Let $k \in \mathbf{Z}^+$. Prove that there exists a positive integer n such that $k|n$ and the only digits in n are 0's and 3's.

17. a) Find a sequence of four distinct real numbers with no decreasing or increasing subsequence of length 3.

b) Find a sequence of nine distinct real numbers with no decreasing or increasing subsequence of length 4.

c) Generalize the results in parts (a) and (b).

d) What do the preceding parts of this exercise tell us about Example 5.49?

18. The 50 members of Nardine's aerobics class line up to get their equipment. Assuming that no two of these people have the same height, show that eight of them (as the line is equipped from first to last) have successive heights that either decrease or increase.

19. For $k, n \in \mathbf{Z}^+$, prove that if $kn + 1$ pigeons occupy n pigeonholes, then at least one pigeonhole has $k + 1$ or more pigeons roosting in it.

20. How many times must we roll a single die in order to get the same score (a) at least twice? (b) at least three times? (c) at least n times, for $n \geq 4$?

21. a) Let $S \subset \mathbf{Z}^+$. What is the smallest value for $|S|$ that guarantees the existence of two elements $x, y \in S$ where x and y have the same remainder upon division by 1000?

b) What is the smallest value of n such that whenever $S \subseteq \mathbf{Z}^+$ and $|S| = n$, then there exist three elements $x, y, z \in S$ where all three have the same remainder upon division by 1000?

c) Write a statement that generalizes the results of parts (a) and (b) and Example 5.42.

22. For $m, n \in \mathbf{Z}^+$, prove that if m pigeons occupy n pigeonholes, then at least one pigeonhole has $\lfloor (m-1)/n \rfloor + 1$ or more pigeons roosting in it.

23. Let $p_1, p_2, \ldots, p_n \in \mathbf{Z}^+$. Prove that if $p_1 + p_2 + \cdots + p_n - n + 1$ pigeons occupy n pigeonholes, then either the first pigeonhole has p_1 or more pigeons roosting in it, or the second pigeonhole has p_2 or more pigeons roosting in it, . . . , or the nth pigeonhole has p_n or more pigeons roosting in it.

24. Given 8 Perl books, 17 Visual BASIC† books, 6 Java books, 12 SQL books, and 20 C++ books, how many of these books must we select to insure that we have 10 books dealing with the same computer language?

5.6 Function Composition and Inverse Functions

When computing with the elements of $\mathbf{Z}$, we find that the (closed binary) operation of addition provides a method for combining two integers, say a and b, into a third integer, namely $a + b$. Furthermore, for each integer c there is a second integer d where $c + d = d + c = 0$, and we call d the additive *inverse* of c. (It is also true that c is the additive *inverse* of d.)

Turning to the elements of $\mathbf{R}$ and the (closed binary) operation of multiplication, we have a method for combining any $r, s \in \mathbf{R}$ into their product rs. And here, for each $t \in \mathbf{R}$, if $t \neq 0$, then there is a real number u such that $ut = tu = 1$. The real number u is called the multiplicative *inverse* of t. (The real number t is also the multiplicative *inverse* of u.)

In this section we first study a method for combining two functions into a single function. Then we develop the concept of the inverse (of a function) for functions with certain properties. To accomplish these objectives, we need the following preliminary ideas.

†Visual BASIC is a trademark of the Microsoft Corporation.

on A, B, (a) $\Longleftrightarrow$ (b). Assuming (b), if f is not one-to-one, then there are elements $a_1, a_2 \in A$, with $a_1 \neq a_2$, but $f(a_1) = f(a_2)$. Then $|A| > |f(A)| = |B|$, contradicting $|A| = |B|$. Conversely, if f is not onto, then $|f(A)| < |B|$. With $|A| = |B|$ we have $|A| > |f(A)|$, and it follows from the pigeonhole principle that f is not one-to-one.

Using Theorem 5.11 we now verify the combinatorial identity introduced in Problem 6 at the start of this chapter. For if $n \in \mathbf{Z}^+$ and $|A| = |B| = n$, there are $n!$ one-to-one functions from A to B and $\sum_{k=0}^{n}(-1)^k\binom{n}{n-k}(n-k)^n$ onto functions from A to B. The equality $n! = \sum_{k=0}^{n}(-1)^k\binom{n}{n-k}(n-k)^n$ is then the numerical equivalent of parts (a) and (b) of Theorem 5.11. [This is also the reason why the diagonal elements $S(n, n)$, $1 \le n \le 8$, shown in Table 5.1 all equal 1.]

EXERCISES 5.6

1. a) For $A = \{1, 2, 3, 4, \ldots, 7\}$, how many bijective functions $f: A \to A$ satisfy $f(1) \neq 1$?

b) Answer part (a) where $A = \{x | x \in \mathbf{Z}^+, 1 \le x \le n\}$, for some fixed $n \in \mathbf{Z}^+$.

2. a) For $A = (-2, 7] \subseteq \mathbf{R}$ define the functions $f, g: A \to \mathbf{R}$ by

$$f(x) = 2x - 4 \quad \text{and} \quad g(x) = \frac{2x^2 - 8}{x + 2}.$$

Verify that $f = g$.

b) Is the result in part (a) affected if we change A to $[-7, 2)$?

3. Let $f, g: \mathbf{R} \to \mathbf{R}$, where $g(x) = 1 - x + x^2$ and $f(x) = ax + b$. If $(g \circ f)(x) = 9x^2 - 9x + 3$, determine a, b.

4. Let $g: \mathbf{N} \to \mathbf{N}$ be defined by $g(n) = 2n$. If $A = \{1, 2, 3, 4\}$ and $f: A \to \mathbf{N}$ is given by $f = \{(1, 2), (2, 3), (3, 5), (4, 7)\}$, find $g \circ f$.

5. If $\mathcal{U}$ is a given universe with (fixed) $S, T \subseteq \mathcal{U}$, define $g: \mathcal{P}(\mathcal{U}) \to \mathcal{P}(\mathcal{U})$ by $g(A) = T \cap (S \cup A)$ for $A \subseteq \mathcal{U}$. Prove that $g^2 = g$.

6. Let $f, g: \mathbf{R} \to \mathbf{R}$ where $f(x) = ax + b$ and $g(x) = cx + d$ for all $x \in \mathbf{R}$, with a, b, c, d real constants. What relationship(s) must be satisfied by a, b, c, d if $(f \circ g)(x) = (g \circ f)(x)$ for all $x \in \mathbf{R}$?

7. Let $f, g, h: \mathbf{Z} \to \mathbf{Z}$ be defined by $f(x) = x - 1$, $g(x) = 3x$,

$$h(x) = \begin{cases} 0, & x \text{ even} \\ 1, & x \text{ odd.} \end{cases}$$

Determine (a) $f \circ g$, $g \circ f$, $g \circ h$, $h \circ g$, $f \circ (g \circ h)$, $(f \circ g) \circ h$; (b) f^2, f^3, g^2, g^3, h^2, h^3, h^{500}.

8. Let $f: A \to B$, $g: B \to C$. Prove that (a) if $g \circ f: A \to C$ is onto, then g is onto; and (b) if $g \circ f: A \to C$ is one-to-one, then f is one-to-one.

9. a) Find the inverse of the function $f: \mathbf{R} \to \mathbf{R}^+$ defined by $f(x) = e^{2x+5}$.

b) Show that $f \circ f^{-1} = 1_{\mathbf{R}^+}$ and $f^{-1} \circ f = 1_{\mathbf{R}}$.

10. For each of the following functions $f: \mathbf{R} \to \mathbf{R}$, determine whether f is invertible, and, if so, determine f^{-1}.

a) $f = \{(x, y) | 2x + 3y = 7\}$

b) $f = \{(x, y) | ax + by = c, b \neq 0\}$

c) $f = \{(x, y) | y = x^3\}$

d) $f = \{(x, y) | y = x^4 + x\}$

11. Prove Theorem 5.9.

12. If $A = \{1, 2, 3, 4, 5, 6, 7\}$, $B = \{2, 4, 6, 8, 10, 12\}$, and $f: A \to B$ where $f = \{(1, 2), (2, 6), (3, 6), (4, 8), (5, 6), (6, 8), (7, 12)\}$, determine the preimage of B_1 under f in each of the following cases.

a) $B_1 = \{2\}$ **b)** $B_1 = \{6\}$
c) $B_1 = \{6, 8\}$ **d)** $B_1 = \{6, 8, 10\}$
e) $B_1 = \{6, 8, 10, 12\}$ **f)** $B_1 = \{10, 12\}$

13. Let $f: \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} x + 7, & x \le 0 \\ -2x + 5, & 0 < x < 3 \\ x - 1, & 3 \le x \end{cases}$$

a) Find $f^{-1}(-10)$, $f^{-1}(0)$, $f^{-1}(4)$, $f^{-1}(6)$, $f^{-1}(7)$, and $f^{-1}(8)$.

b) Determine the preimage under f for each of the intervals (i) $[-5, -1]$; (ii) $[-5, 0]$; (iii) $[-2, 4]$; (iv) $(5, 10)$; and (v) $[11, 17)$.

14. Let $f: \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = x^2$. For each of the following subsets B of $\mathbf{R}$, find $f^{-1}(B)$.

a) $B = \{0, 1\}$ **b)** $B = \{-1, 0, 1\}$
c) $B = [0, 1]$ **d)** $B = [0, 1)$
e) $B = [0, 4]$ **f)** $B = (0, 1] \cup (4, 9)$

15. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{6, 7, 8, 9, 10, 11, 12\}$. How many functions $f: A \to B$ are such that $f^{-1}(\{6, 7, 8\}) = \{1, 2\}$?

16. Let $f: \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = \lfloor x \rfloor$, the greatest integer in x. Find $f^{-1}(B)$ for each of the following subsets B of $\mathbf{R}$.

a) $B = \{0, 1\}$
b) $B = \{-1, 0, 1\}$
c) $B = [0, 1)$
d) $B = [0, 2)$
e) $B = [-1, 2]$
f) $B = [-1, 0) \cup (1, 3]$

17. Let $f, g: \mathbf{Z}^+ \to \mathbf{Z}^+$ where for all $x \in \mathbf{Z}^+$, $f(x) = x + 1$ and $g(x) = \max\{1, x - 1\}$, the maximum of 1 and $x - 1$.

a) What is the range of f?
b) Is f an onto function?
c) Is the function f one-to-one?
d) What is the range of g?
e) Is g an onto function?
f) Is the function g one-to-one?
g) Show that $g \circ f = 1_{\mathbf{Z}^+}$.
h) Determine $(f \circ g)(x)$ for $x = 2, 3, 4, 7, 12$, and 25.
i) Do the answers for parts (b), (g), and (h) contradict the result in Theorem 5.8?

18. Let f, g, h denote the following closed binary operations on $\mathcal{P}(\mathbf{Z}^+)$. For $A, B \subseteq \mathbf{Z}^+$, $f(A, B) = A \cap B$, $g(A, B) = A \cup B$, $h(A, B) = A \triangle B$.

a) Are any of the functions one-to-one?
b) Are any of f, g, and h onto functions?
c) Is any one of the given functions invertible?
d) Are any of the following sets infinite?

(1) $f^{-1}(\emptyset)$ (2) $g^{-1}(\emptyset)$
(3) $h^{-1}(\emptyset)$ (4) $f^{-1}(\{1\})$
(5) $g^{-1}(\{2\})$ (6) $h^{-1}(\{3\})$
(7) $f^{-1}(\{4, 7\})$ (8) $g^{-1}(\{8, 12\})$
(9) $h^{-1}(\{5, 9\})$

e) Determine the number of elements in each of the finite sets in part (d).

19. Prove parts (a) and (c) of Theorem 5.10.

20. a) Give an example of a function $f: \mathbf{Z} \to \mathbf{Z}$ where (i) f is one-to-one but not onto; and (ii) f is onto but not one-to-one.
b) Do the examples in part (a) contradict Theorem 5.11?

21. Let $f: \mathbf{Z} \to \mathbf{N}$ be defined by

$$f(x) = \begin{cases} 2x - 1, & \text{if } x > 0 \\ -2x, & \text{for } x \leq 0. \end{cases}$$

a) Prove that f is one-to-one and onto.
b) Determine f^{-1}.

22. If $|A| = |B| = 5$, how many functions $f: A \to B$ are invertible?

23. Let $f, g, h, k: \mathbf{N} \to \mathbf{N}$ where $f(n) = 3n$, $g(n) = \lfloor n/3 \rfloor$, $h(n) = \lfloor (n + 1)/3 \rfloor$, and $k(n) = \lfloor (n + 2)/3 \rfloor$, for each $n \in \mathbf{N}$. (a) For each $n \in \mathbf{N}$ what are $(g \circ f)(n)$, $(h \circ f)(n)$, and $(k \circ f)(n)$? (b) Do the results in part (a) contradict Theorem 5.7?

5.7 Computational Complexity†

In Section 4.4 we introduced the concept of an algorithm, following the examples set forth by the division algorithm (of Section 4.3) and the Euclidean algorithm (of Section 4.4). At that time we were concerned with certain properties of a general algorithm:

- The precision of the individual step-by-step instructions
- The input provided to the algorithm, and the output the algorithm then provides
- The ability of the algorithm to solve a certain type of problem, not just specific instances of the problem
- The uniqueness of the intermediate and final results, based on the input

† The material in Sections 5.7 and 5.8 may be skipped at this point. It will not be used very much until Chapter 10. The only place where this material appears before Chapter 10 is in Example 7.13, but that example can be omitted without any loss of continuity.