Constructing Parametric Triangular Patches with Boundary Conditions

Hui Liu
Department of Computer Science and Technology
Shandong University,
Jinan,250061, China
liuh_lh@126.com

Caiming Zhang
Department of Computer Science and Technology
Shandong University,
Jinan,250061, China
czhang@sdu.edu.cn

Fuhua (Frank) Cheng
Department of Computer Science
University of Kentucky,
Lexington, KY 40506,USA
cheng@cs.uky.edu

Abstract

The problem of constructing a parametric triangular patch to smoothly connect three surface patches is studied. Usually, these surface patches are defined on different parameter spaces. Therefore, it is necessary to define interpolation conditions, with values from the given surface patches, on the boundary of the triangular patch that can ensure smooth transition between different parameter spaces. In this paper we present a new method to define boundary conditions. Boundary conditions defined by the new method have the same parameter space if the three given surface patches can be converted into the same form through affine transformation. Consequently,any of the classic methods for constructing functional triangular patches can be used directly to construct a parametric triangular patch to connect the given surface patches with G^1 continuity. The resulting parametric triangular patch preserves precision of the adopted classic method.

1 Introduction

Construction of surfaces plays an important role in computer aided geometric design (CAGD), free-form surface modeling and computer graphics (CG). To make the process of constructing complex surfaces simple, piecewise techniques are frequently used, with four-sided and triangular patches being the most popular choices. This paper studies the problem of boundary condition determination in the process of constructing parametric triangular patches to smoothly connect three given surface patches. These surface patches can be of any form. Therefore, the problem addressed here can also be viewed as an infinite interpolation on triagles.

Infinite interpolation on triangles was first studied by Barnhill, Birkhoff and Gordon [1], and a curved triangular patch that interpolates boundary conditions of any form was proposed. The triangular patch is constructed using the *Boolean sum scheme*. Gregory [2] used the *convex combination method* to construct a triangular patch. The triangular patch is formed by the convex combination of three interpolation operators, each of which satisfies the interpolation conditions on two sides of a triangle. The idea [2] was further extended in papers [3, 4]. Nielson [5] presented a *side-vertex method* to construct a curved triangular patch using combination of three interpolation operators, each satisfying the given boundary conditions at a vertex and its opposite side. Hagen [6] extended Nielson's approach to construct *geometric patches*. These results have been generalized to triangular patches with first and second order geometric continuity [7, 8]. The problem of constructing non-four-sided patches including curved triangular patches was also studied in [9, 10]. In [11] a method to construct a curved triangular patch by combining four interpolation operators: an *interior interpolation operator* and three *side-vertex operators* [5] is presented. The constructed triangular patch reproduces polynomial surfaces of degree four. Another method proposed recently [12] constructs a triangular patch by a *basic approximation operator* and an *interpolation operator*. The constructed triangular patch satisfies C^1 boundary condition and reproduces polynomial surfaces of degree five.

The above methods all work on the assumption that the interpolation conditions on the boundary of the triangle are defined on the same parameter space. In practice, however, this is usually not the case. It is therefore necessary to have a method to determine suitable interpolation conditions so that the methods [1]-[12] can be used directly to construct parametric triangular patches. In [13], a method is presented to construct the cross-boundary conditions. The constructed cross-boundary conditions have suitable magnitudes, but not suitable directions on the boundary of the triangle. This paper overcomes this problem by presenting a simple but efficient method to construct cross-boundary conditions which have both suitable magnitudes and directions. The combination of the new method and any of the classic functional triangular patch construction methods [1]-[12] can be used to construct a G^1 parametric triangular patch to connect three given surface patches. The constructed parametric triangular patch has the same interpolation precision as the adopted classic methods [1]-[12].

2 Problem description

Suppose $\boldsymbol{P}_i(s_i,t_i) = (x_i(s_i,t_i), y_i(s_i,t_i), z_i(s_i,t_i))$, $(0 \le s_i, t_i \le 1)$, $i = 1,2,3$, are three given surface patches, defined on different s_it_i-parametric planes. The three patches are of any form. The three patches meet in the way shown in Figure 1. The goal is to construct a triangular patch $\boldsymbol{P}_T(s,t)$ to connect the three patches $\boldsymbol{P}_i(s_i,t_i)$, $i = 1,2,3$, with G^1 continuity. $\boldsymbol{P}_T(s,t)$ and $\boldsymbol{P}_i(s_i,t_i)$, $i = 1,2,3$, being G^1 continuous means that they have a common boundary and the normal vectors of them on the common boundary have the same direction.

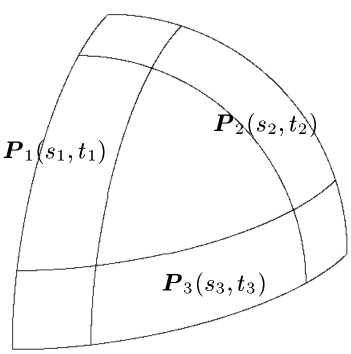

Figure 1. Three surfaces meet

If these three patches are defined on the same parametric st-plane, then the methods for constructing functional triangular patches can be used directly to construct a parametric triangular patch to connect these patches with C^1 continuity. In most applications of CAGD, CG and related areas, however, these three patches usually are not defined on the same parameter space. In this case, one needs to define C^1 boundary conditions by the three patches so that the constructed parametric triangular patch can smoothly connect these patches with a "visually pleasing shape" suggested by these three patches. After the C^1 boundary conditions are defined, the functional methods of constructing triangular patches can be used to construct parameter triangular patch directly. As $\boldsymbol{P}_T(s,t)$ and $\boldsymbol{P}_i(s_i,t_i)$, $i = 1,2,3$, are defined on different parameter spaces, $\boldsymbol{P}_T(s,t)$, satisfying C^1 boundary conditions, will connect these three patches with G^1 continuity.

Let T be an equilateral triangle with vertices $\boldsymbol{v}_1 = (0,0)$, $\boldsymbol{v}_2 = (1,0)$ and $\boldsymbol{v}_3 = (1/2, \sqrt{3}/2)$ in the st-parametric space, e_i denote the opposite side of $\boldsymbol{v}_i$ and τ_i is the unit outward normal vector of e_i, as shown in Figure 2. Let σ_1 denote the unit vector from $\boldsymbol{v}_2$ to $\boldsymbol{v}_3$. σ_2 and σ_3 are defined similarly. The sides e_i, $i = 1,2,3$, can be parameterized as follows:

$$\begin{aligned} \boldsymbol{e}_1(u) &= (1-u)\boldsymbol{v}_2 + u\boldsymbol{v}_3, \\ \boldsymbol{e}_2(u) &= (1-u)\boldsymbol{v}_1 + u\boldsymbol{v}_3, \qquad 0 \le u \le 1 \\ \boldsymbol{e}_3(u) &= (1-u)\boldsymbol{v}_1 + u\boldsymbol{v}_2, \end{aligned} \tag{1}$$

The parametric triangular patch $\boldsymbol{P}_T(s,t)$ to be constructed will be defined on the equilateral triangle T, as shown in Figure 2. On the three sides of T, the boundary curve and cross-boundary slope conditions given by the three surfaces, $\boldsymbol{P}_i(s_i,t_i)$, $i = 1,2,3$ are as follows

$$\boldsymbol{P}_i(\boldsymbol{e}_i(u)), \ \frac{\partial \boldsymbol{P}_i}{\partial s_i}(\boldsymbol{e}_i(u)), \quad i = 1,2,3 \tag{2}$$

where $\boldsymbol{e}_i(u)$'s are defined in Eq. (1), $\boldsymbol{P}_i(\boldsymbol{e}_i(u))$ and $\frac{\partial \boldsymbol{P}_i}{\partial s_i}(\boldsymbol{e}_i(u))$ denote the boundary value and the cross-boundary slope of $\boldsymbol{P}_i(s_i,t_i)$ on the side e_i, respectively.

As the boundary conditions (2) cannot be used directly to construct the triangular patch on T, we will use them to define the new boundary conditions. Let the new boundary conditions be

$$\boldsymbol{P}_T(\boldsymbol{e}_i(u)), \qquad \frac{\partial \boldsymbol{P}_T}{\partial \tau_i}(\boldsymbol{e}_i(u)), \quad i = 1,2,3. \tag{3}$$

The new boundary conditions (3) should be defined in a way so that if the three patches $\boldsymbol{P}_i(s_i,t_i)$,$i = 1,2,3$ are defined by the same surface $\boldsymbol{P}(s,t)$, but with different parameter spaces, then $\boldsymbol{P}_T(\boldsymbol{e}_i(u))$, $\frac{\partial \boldsymbol{P}_T}{\partial \tau_i}(\boldsymbol{e}_i(u))$, $i = 1,2,3$ on the three sides of T in Figure 2 can be defined by $\boldsymbol{P}(s,t)$, i.e., by

$$\left.\begin{aligned} &\boldsymbol{P}_T(\boldsymbol{e}_i(u)) = \boldsymbol{P}(\boldsymbol{e}_i(u)), \\ &\frac{\partial \boldsymbol{P}_T}{\partial \tau_i}(\boldsymbol{e}_i(u)) = \frac{\partial \boldsymbol{P}}{\partial \tau_i}(\boldsymbol{e}_i(u)), \end{aligned}\right. \ , \ i = 1,2,3 \tag{4}$$

3 Constructing the boundary Conditions

We show how to determine $\boldsymbol{P}_T(\boldsymbol{e}_i(u))$, $\frac{\partial \boldsymbol{P}_T}{\partial \tau_i}(\boldsymbol{e}_i(u))$, $i = 1,2,3$, in this section. As shown in Figure 3, suppose that the surface patch $\boldsymbol{P}_1(s_1,t_1)$ is defined on the parallelogram region $\boldsymbol{v}_2\boldsymbol{v}_3\boldsymbol{v}_4\boldsymbol{v}_5$, $\boldsymbol{P}_2(s_2,t_2)$ and $\boldsymbol{P}_3(s_3,t_3)$ are similarly defined. The $\boldsymbol{P}_T(s,t)$ and $\boldsymbol{P}_i(s_i,t_i)$ are G^1 continuous on the common boundary, thus $\boldsymbol{P}_T(\boldsymbol{e}_i(u))$, $\frac{\partial \boldsymbol{P}_T}{\partial \tau_i}(\boldsymbol{e}_i(u))$, $i = 1,2,3$ can be defined by $\boldsymbol{P}_i(s_i,t_i)$, $i = 1,2,3$ as follows:

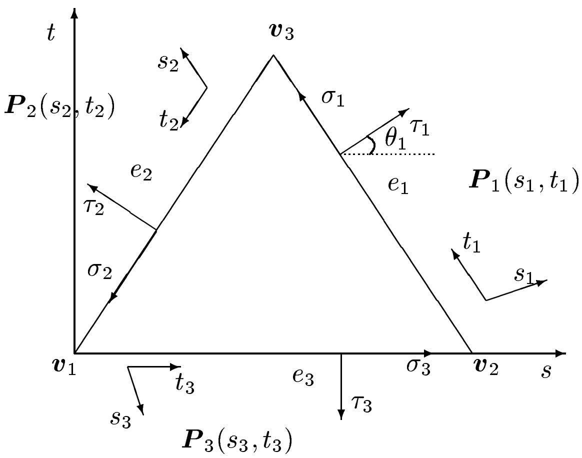

Figure 2. Three patches meet on T.

$$\begin{aligned}
&\boldsymbol{P}_T(\boldsymbol{e}_i(u)) = \boldsymbol{P}_i(\boldsymbol{e}_i(u)), \\
&\frac{\partial \boldsymbol{P}_T}{\partial \tau_i}(\boldsymbol{e}_i(u)) = \alpha_i(\boldsymbol{e}_i(u))\frac{\partial \boldsymbol{P}_i}{\partial s_i}(\boldsymbol{e}_i(u)) + \\
&\beta_i(\boldsymbol{e}_i(u))\frac{\partial \boldsymbol{P}_i}{\partial t_i}(\boldsymbol{e}_i(u)),\ i = 1, 2, 3
\end{aligned} \tag{5}$$

where $\alpha_i(\boldsymbol{e}_i(u))$ and $\beta_i(\boldsymbol{e}_i(u))$ are functions of u to be constructed, respectively.

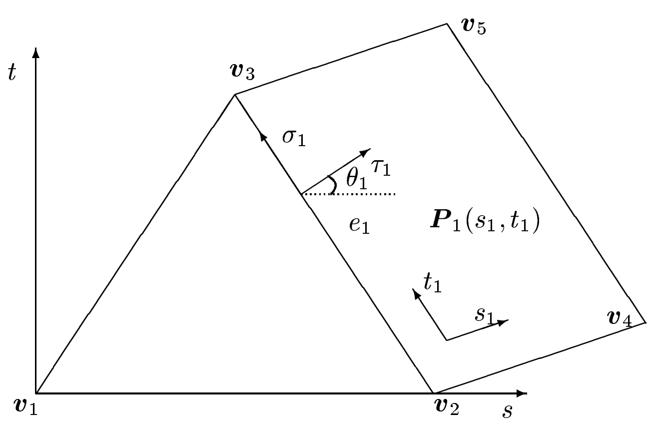

Figure 3. Three patches meet on T.

Now, constructing the boundary conditions becomes a problem of defining the functions $\alpha_i(\boldsymbol{e}_i(u))$ and $\beta_i(\boldsymbol{e}_i(u))$ (5), $i = 1, 2, 3$. For simplicity, we shall show the construction process of $\alpha_1(\boldsymbol{e}_1(u))$ and $\beta_1(\boldsymbol{e}_1(u))$ only. The $\alpha_i(\boldsymbol{e}_i(u))$ and $\beta_i(\boldsymbol{e}_i(u))$, $i = 2, 3$ can be constructed similarly.

As vectors σ_1 and t_1 are same, see Figure 3, so vectors τ_1 and t_1 are orthonormal, thus $\frac{\partial \boldsymbol{P}_T}{\partial \tau_1}(\boldsymbol{e}_1(u))$ and $\frac{\partial \boldsymbol{P}_T}{\partial t_1}(\boldsymbol{e}_1(u))$ satisfy

$$\langle \frac{\partial \boldsymbol{P}_T}{\partial \tau_1}(\boldsymbol{e}_1(u)) \cdot \frac{\partial \boldsymbol{P}_T}{\partial t_1}(\boldsymbol{e}_1(u))\rangle = 0$$

where $\langle \boldsymbol{a} \cdot \boldsymbol{b}\rangle$ denotes the dot product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. It follows from (5) that

$$A_1\alpha_1(\boldsymbol{e}_1(u)) + B_1\beta_1(\boldsymbol{e}_1(u)) = 0 \tag{6}$$

where

$$\begin{aligned}
A_1 &= \langle \frac{\partial \boldsymbol{P}_1}{\partial s_1}(\boldsymbol{e}_1(u)) \cdot \frac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{e}_1(u))\rangle, \\
B_1 &= \langle \frac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{e}_1(u)) \cdot \frac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{e}_1(u))\rangle
\end{aligned}$$

If s_1 and t_1 are orthonormal,
$A_1 = \langle \frac{\partial \boldsymbol{P}_1}{\partial s_1}(\boldsymbol{e}_1(u)) \cdot \frac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{e}_1(u))\rangle = 0$,
thus the function relation between $\alpha_1(\boldsymbol{e}_1(u))$ and $\beta_1(\boldsymbol{e}_1(u))$ is taken as

$$\beta_1(\boldsymbol{e}_1(u)) = -A_1\alpha_1(\boldsymbol{e}_1(u))/B_1 \tag{7}$$

The Eq. (7) shows that if $\beta_1(\boldsymbol{e}_1(u))$ is defined, then $\alpha_1(\boldsymbol{e}_1(u))$ is defined. In the following we show how to construct $\beta_1(\boldsymbol{e}_1(u))$. We first determine the values of $\alpha_1(\boldsymbol{e}_1(u))$ and $\beta_1(\boldsymbol{e}_1(u))$ at points $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$, respectively. At point $\boldsymbol{v}_2$, we have

$$\frac{\partial \boldsymbol{P}_T}{\partial \tau_1}(\boldsymbol{v}_2) = \alpha_1(v_2)\frac{\partial \boldsymbol{P}_1}{\partial s_1}(\boldsymbol{v}_2) + \beta_1(v_2)\frac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{v}_2). \tag{8}$$

The angle θ_1 between vectors τ_1 and t_3 is 30^o, thus

$$\frac{\partial \boldsymbol{P}_3}{\partial t_3}(\boldsymbol{v}_2) = \frac{\sqrt{3}}{2}\frac{\partial \boldsymbol{P}_T}{\partial \tau_1}(\boldsymbol{v}_2) - \frac{1}{2}\frac{\partial \boldsymbol{P}_T}{\partial \sigma_1}(\boldsymbol{v}_2).$$

From

$$\frac{\partial \boldsymbol{P}_T}{\partial \sigma_1}(\boldsymbol{v}_2) = \frac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{v}_2),$$

we have

$$\frac{\partial \boldsymbol{P}_T}{\partial \tau_1}(\boldsymbol{v}_2) = \frac{2\sqrt{3}}{3}\frac{\partial \boldsymbol{P}_3}{\partial t_3}(\boldsymbol{v}_2) + \frac{\sqrt{3}}{3}\frac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{v}_2). \tag{9}$$

It follows from Eq. (8) and Eq. (9) that $\alpha_1(v_2)$ and $\beta_1(v_2)$ in Eq. (5), denoted α_1^0 and β_1^0, can be determined by the following equations.

$$\begin{aligned}
&\langle \frac{\partial \boldsymbol{P}_1}{\partial s_1}(\boldsymbol{v}_2) \cdot \frac{\partial \boldsymbol{P}_1}{\partial s_1}(\boldsymbol{v}_2)\rangle\alpha_1^0 + \langle \frac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{v}_2) \cdot \frac{\partial \boldsymbol{P}_1}{\partial s_1}(\boldsymbol{v}_2)\rangle\beta_1^0 = \\
&\langle \frac{\partial \boldsymbol{P}_T}{\partial \tau_1}(\boldsymbol{v}_2) \cdot \frac{\partial \boldsymbol{P}_1}{\partial s_1}(\boldsymbol{v}_2)\rangle, \\
&\langle \frac{\partial \boldsymbol{P}_1}{\partial s_1}(\boldsymbol{v}_2) \cdot \frac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{v}_2)\rangle\alpha_1^0 + \langle \frac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{v}_2) \cdot \frac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{v}_2)\rangle\beta_1^0 = 0
\end{aligned} \tag{10}$$

On the other hand, at $\boldsymbol{v}_3$ we have

$$\frac{\partial \boldsymbol{P}_T}{\partial \tau_1}(\boldsymbol{v}_3) = \alpha_1(v_3)\frac{\partial \boldsymbol{P}_1}{\partial s_1}(\boldsymbol{v}_3) + \beta_1(v_3)\frac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{v}_3),$$
$$\frac{\partial \boldsymbol{P}_T}{\partial \tau_1}(\boldsymbol{v_3}) = -\frac{2\sqrt{3}}{3}\frac{\partial \boldsymbol{P}_2}{\partial t_2}(\boldsymbol{v}_3) - \frac{\sqrt{3}}{3}\frac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{v}_3). \tag{11}$$

Thus $\alpha_1(v_3)$ and $\beta_1(v_3)$ in Eq. (5), denoted α_1^1 and β_1^1, can be determined by the following equations.

$$\langle\frac{\partial \boldsymbol{P}_1}{\partial s_1}(\boldsymbol{v}_3)\cdot\frac{\partial \boldsymbol{P}_1}{\partial s_1}(\boldsymbol{v}_3)\rangle\alpha_1^1 + \langle\frac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{v}_3)\cdot\frac{\partial \boldsymbol{P}_1}{\partial s_1}(\boldsymbol{v}_3)\rangle\beta_1^1 = \langle\frac{\partial \boldsymbol{P}_T}{\partial \tau_1}(\boldsymbol{v}_3)\cdot\frac{\partial \boldsymbol{P}_1}{\partial s_1}(\boldsymbol{v_3})\rangle,$$
$$\langle\frac{\partial \boldsymbol{P}_1}{\partial s_1}(\boldsymbol{v}_3)\cdot\frac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{v}_3)\rangle\alpha_1^1 + \langle\frac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{v}_3)\cdot\frac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{v}_3)\rangle\beta_1^1 = 0. \tag{12}$$

For $\alpha(\boldsymbol{e}_1(u))$, two values α_1^0 and α_1^1 are computed, thus a suitable choice is that $\alpha_1(\boldsymbol{e}_1(u))$ is defined by a linear interpolation as follows:

$$\alpha_1(\boldsymbol{e}_1(u)) = (1-u)\alpha_1^0 + u\alpha_1^1 \quad 0 \le u \le 1 \tag{13}$$

where α_1^0 and α_1^1 are defined by (10) and (12).

Based on (7) and (13), $\alpha_1(\boldsymbol{e}_1(u))$ and $\beta_1(\boldsymbol{e}_1(u))$ are defined by

$$\begin{aligned}\alpha_1(\boldsymbol{e}_1(u)) &= (1-u)\alpha_1^0 + u\alpha_1^1\\ \beta_1(\boldsymbol{e}_1(u)) &= -A_1\alpha_1(\boldsymbol{e}_1(u))/B_1.\end{aligned}, \quad 0 \le u \le 1 \tag{14}$$

where A_1 and B_1 are defined by (6).

Similarly, one can define $\alpha_i(\boldsymbol{e}_i(u))$ and $\beta_i(\boldsymbol{e}_i(u))$ for $i = 2, 3$ as follows:

$$\begin{aligned}\alpha_2(\boldsymbol{e}_2(u)) &= (1-u)\alpha_2^0 + u\alpha_2^1\\ \beta_2(\boldsymbol{e}_2(u)) &= -A_2\alpha_2(\boldsymbol{e}_2(u))/B_2.\\ \alpha_3(\boldsymbol{e}_3(u)) &= (1-u)\alpha_3^0 + u\alpha_3^1\\ \beta_3(\boldsymbol{e}_3(u)) &= -A_3\alpha_3(\boldsymbol{e}_3(u))/B_3.\end{aligned} \quad 0 \le u \le 1. \tag{15}$$

The above construction process of C^1 boundary conditions shows that when the methods for constructing C^1 functional triangular patch are directly applied to the boundary conditions in Eq. (5), a parameter patch $\boldsymbol{P}_T(s,t)$ is constructed, which connects $\boldsymbol{P}_i(s_i,t_i)$, $i = 1, 2, 3$ with G^1 continuity and smooth shape.

4 Discussion

In this section, we will show that the cross-boundary slopes defined by Eqs. (5), (14) and (15) are well defined. To do this, one only needs to prove that if the three surfaces $\boldsymbol{P}_i(s_i,t_i)$, $i = 1, 2, 3$, are defined by the same surface $\boldsymbol{P}(s,t)$ but in different forms, which are formed by applying affine transformations on $\boldsymbol{P}(s,t)$, then the new boundary conditions are defined by (4), i.e., by $\boldsymbol{P}(s,t)$. This means that if a method reproduces polynomials of degree n when it is used to construct functional triangular patches, then when it is used with the boundary conditions (5) to construct a parametric triangular patch $\boldsymbol{P}_T(s,t)$, $\boldsymbol{P}_T(s,t)$ will reproduce parametric polynomials of degree n.

Theorem 1 *If surface patches $\boldsymbol{P}_i(s_i,t_i)$, $i = 1, 2, 3$, are defined by the same surface $\boldsymbol{P}(s,t)$, i.e, $\boldsymbol{P}(\tau_1,\sigma_1)$, and the transformations from coordinate system st to coordinate system s_it_i are affine, then there exist unique constants c_i and d_i satisfying the following conditions*

$$\begin{aligned}\alpha_i &= 1/c_i,\\ \beta_i &= -d_i/c_i\end{aligned} \tag{16}$$

where α_i and β_i satisfy $\alpha_i(\boldsymbol{e}_i(u)) = \alpha_i$ and $\beta_i(\boldsymbol{e}_i(u)) = \beta_i$, which means $\alpha_i(\boldsymbol{e}_i(u))$ and $\beta_i(\boldsymbol{e}_i(u))$ in Eq. (5) are constants in this case.

Proof Only the case $i = 1$ will be considered. The other two cases can be handled similarly. Let $\boldsymbol{V}$ be any point in parametric space, in $\tau_1\sigma_1$ and s_1t_1 coordinate systems, the coordinates of $\boldsymbol{V}$ be (τ_1,σ_1) and (s_1,t_1), respectively. As the transformation from coordinate system st to coordinate system s_1t_1 is affine, vectors $\vec{\tau}_1$ and $\vec{t}_1$ are same, as shown in Figure 3, the relationship between (τ_1,σ_1) and (s_1,t_1) can be written as

$$\begin{aligned}\tau_1 &= c_1 s_1,\\ \sigma_1 &= d_1 s_1 + t_1.\end{aligned} \tag{17}$$

As $\boldsymbol{P}_i(s_1,t_1)$ is defined by $\boldsymbol{P}(\tau_1,\sigma_1)$, it follows from Eq. (17) that $\boldsymbol{P}_1(s_1,t_1)$ can be expressed as

$$\boldsymbol{P}_1(s_1,t_1) = \boldsymbol{P}(c_1s_1, d_1s_1 + t_1) = \boldsymbol{P}(\tau_1,\sigma_1).$$

Now

$$\begin{aligned}\frac{\partial \boldsymbol{P}_1(s_1,t_1)}{\partial s_1} &= c_1\frac{\partial \boldsymbol{P}(\tau_1,\sigma_1)}{\partial \tau_1} + d_1\frac{\partial \boldsymbol{P}(\tau_1,\sigma_1)}{\partial \sigma_1},\\ \frac{\partial \boldsymbol{P}_1(s_1,t_1)}{\partial t_1} &= \frac{\partial \boldsymbol{P}(\tau_1,\sigma_1)}{\partial \sigma_1}\end{aligned}$$

Thus

$$\begin{aligned}\frac{\partial \boldsymbol{P}(\tau_1,\sigma_1)}{\partial \tau_1} &= \frac{1}{c_1}\frac{\partial \boldsymbol{P}(s_1,t_1)}{\partial s_1} - \frac{d_1}{c_1}\frac{\partial \boldsymbol{P}(s_1,t_1)}{\partial t_1}.\\ \frac{\partial \boldsymbol{P}(\tau_1,\sigma_1)}{\partial \sigma_1} &= \frac{\partial \boldsymbol{P}(s_1,t_1)}{\partial t_1}\end{aligned}$$

and this completes the proof of the theorem.

In CAGD and CG applications, the curves and surfaces are generally defined on normalized domains, [0,1] for curves and $[0,1]\times[0,1]$ for surfaces. In most cases, the domains of curves and surfaces are normalized by affine transformations, thus in Theorem 1, that the transformation froms $\boldsymbol{P}(s,t)$ to $\boldsymbol{P}_i(s_i,t_i)$, $i = 1, 2, 3$, are restricted as affine transformations is reasonable. Theorem 1 shows that if surfaces $\boldsymbol{P}_i(s_i,t_i)$, $i = 1, 2, 3$, are defined by the same surface, then α_1^0 and β_1^0 in Eq.(10) and α_1^1 and β_1^1 in Eq. (12) satisfy $\alpha_1^0 = \alpha_1^1$ and $\beta_1^0 = \beta_1^1$, so the functions $\alpha_i(\boldsymbol{e}_i(u))$ and $\beta_i(\boldsymbol{e}_i(u))$ in Eq.(5), $i = 1, 2, 3$, are uniquely determined, i.e., determined by Eq.(4). Consequently, the interpolation conditions are determined uniquely, thus the triangular patch to be constructed is determined uniquely. Therefore the following theorem follows.

Theorem 2 *If the method of constructing functional triangular patch reproduces polynomials of degree n, and the method is directly applied on the interpolation conditions in Eq.(5), then the constructed parametric triangular patch $\boldsymbol{P}_T(s,t)$ reproduces parametric polynomials of degree n.*

5 Experiment

Experiment results presented in this section are carried out by constructing a parametric triangular patch to connect three patches. The first experiment is to construct a triangular patch to connect three surfaces, $\boldsymbol{P}_i(s_i, t_i)$, $(0 \leq s_i, t_i \leq 1)$, $i = 1, 2, 3$, as shown in Figure 4. The triangular patches are produced by Nielson's method [5]. In Figure 5, the triangular patch in (a) is produced by directly applying Nielson's method [5] on the boundary curves and cross-boundary slopes defined by the three rectangle patches. The triangular patches in (b) and (c) are produced by using the method presented in [13] and the technique presented in this paper, respectively, to redefine the cross-boundary slopes taken from the three given rectangular patches, then applying Nielson's method [5] on the boundary curves and the redefined cross-boundary slopes. In Figure 5, some portions of the surfaces on the common boundary of the triangular patch with the three rectangular patches are visually not very smooth. This is the result of Mach band phenomenon. Figures 5 show that surfaces in (c) have less Mach band phenomenon than those of (b).

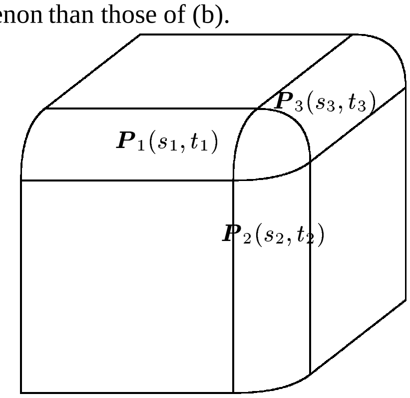

Figure 4. Three surfaces meet

Highlight lines [14] have been proved to be effective tool in assessing the quality of a surface. In Figure 6, the highlight line model is used to compare the above three methods. The figures in Figure 6 are highlight lines of the horizontal

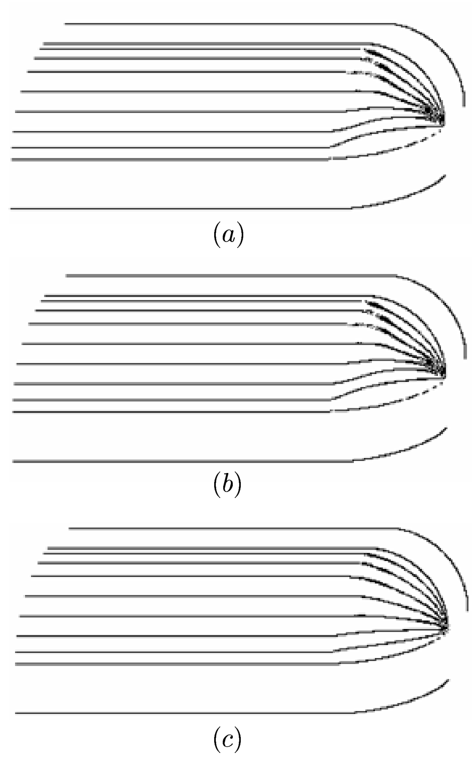

Figure 6. Example 4

fillets of the surfaces in Figure 5. The figures in Figure 6 show that the new method gets better results than the other two methods.

The second experiment is to test the new method using the two functions presented by Franke [15] are used in the comparison process. They are

$$F_4(x,y) = 5.2exp[-81((x-0.5)^2 + (y-0.5)^2)/16]/3,$$
$$F_5(x,y) = 5.2exp[-81((x-0.5)^2 + (y-0.5)^2)/4]/3$$

The set of data points (including 33 points) presented in [15] is used to produce triangles for comparison. The triangulation of the data set is performed using the max-min criterion proposed by Lawson [16] (see Figure 7).

The new method is compared by applying it to Nielson's method (Theorem 3.1 of [5]), ZC's method[11] and ZJY's method[12]. Nielson's method, ZC's method and ZJY's method have the polynomial interpolation precision of degree three, four and five, respectively. The new method is tested by expressing the two functions $F_4(x,y)$ and $F_5(x,y)$ above by parametric form, $P_4(u,v) = (x(u,v), y(u,v), F_4(u,v))$ and $P_5(u,v) =$

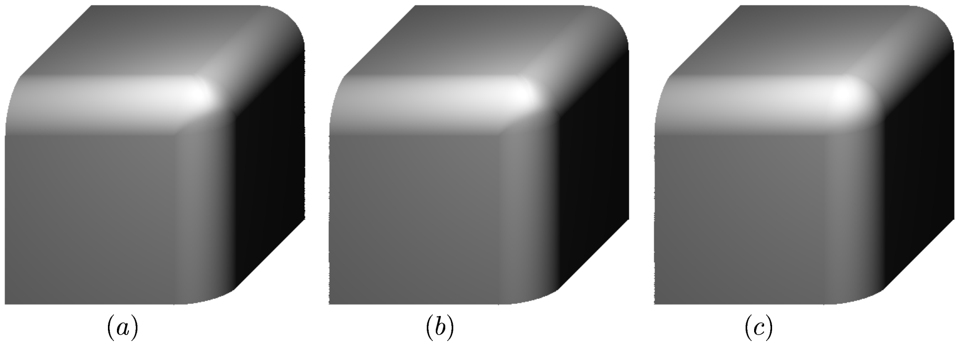

Figure 5. Example 2

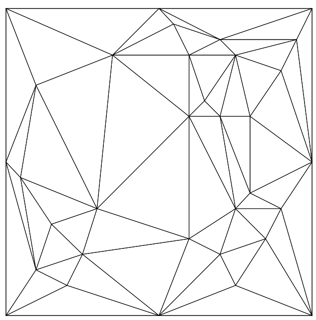

Figure 7. Triangulation of 33 points.

$(x(u,v), y(u,v), F_5(u,v))$, which are defined by

$$
\begin{aligned}
&F_4(u,v) = 5.2exp[-81((u-0.5)^2 + (v-0.5)^2)/16]/3,\\
&F_5(u,v) = 5.2exp[-81((u-0.5)^2 + (v-0.5)^2)/4]/3,\\
&x(u,v) = u,\\
&y(u,v) = v
\end{aligned}
\tag{18}
$$

Two surfaces defined by (18) are used to define the boundary curves and cross-boundary slopes on the sides of the triangles in Figure 7, for each side of the triangles, the cross-boundary slopes defined by (18) is multiplied by 0.8 to simulate it to be of any form. For the interpolation conditions on the triangles, the surfaces by directly using the three methods,respectively, are shown in Figures 8-9. While for the interpolation conditions on the triangles, the cross-boundary conditions are first redefined by the new method, then the surfaces by directly using the three methods,respectively, are shown in Figures 10-11.

6 Conclusions

A new method that uses functional triangular patch construction methods to construct parametric triangular patches is presented. The new method improves previous methods in both surface shape and surface quality. This is testified

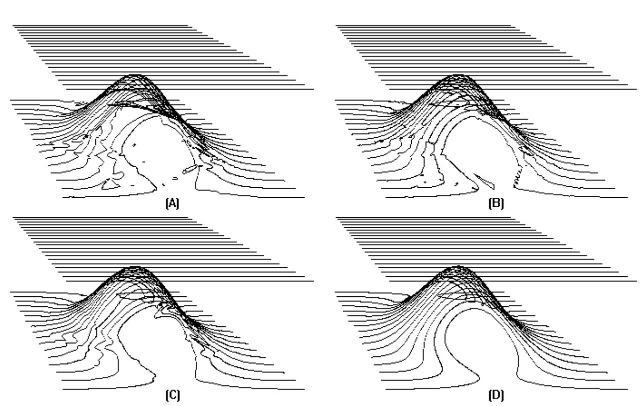

Figure 8. (A) Nielson's method, (B) ZC's method, (C) ZJY's method, (D) $F_4(x,y)$**.**

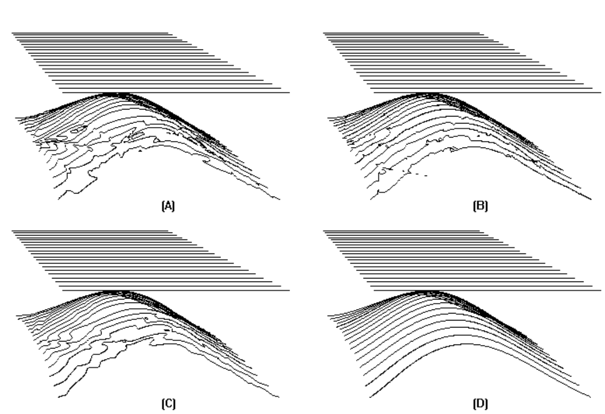

Figure 9. (A) Nielson's method, (B) ZC's method, (C) ZJY's method, (D) $F_5(x,y)$**.**

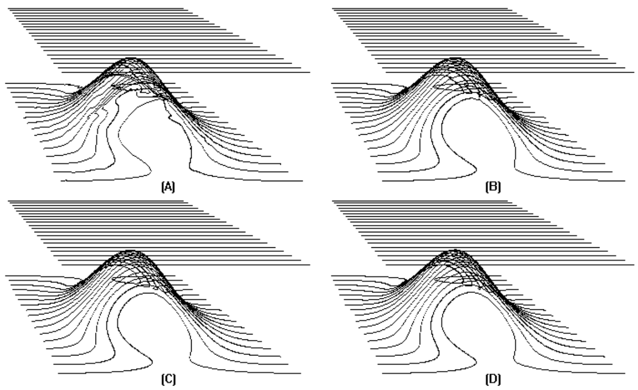

Figure 10. (A) Nielson's method, (B) ZC's method, (C) ZJY's method, (D) $P_4(u,v)$.

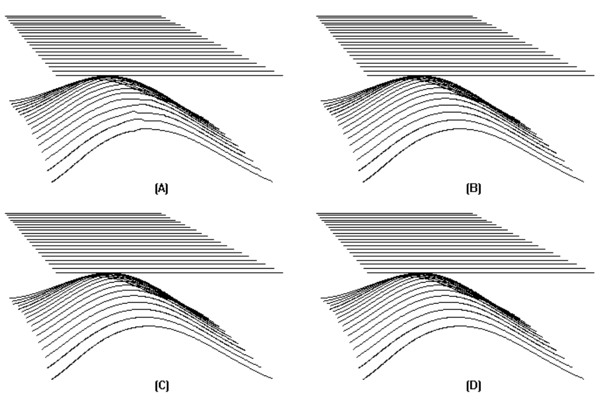

Figure 11. (A) Nielson's method, (B) ZC's method, (C) ZJY's method, (D) $P_5(u,v)$.

by examining Mach band effect and highlight line models of the resulting surface patches. The key in achieving the improvement is a technique to define the cross-boundary conditions. The resulting cross-boundary conditions have not only suitable magnitudes but suitable directions as well.

With the new method, one can directly apply any of the classic functional triangular patch construction methods to construct a C^1 parametric triangular patch to smoothly connect three surface patches. The new method preserves precision of the adopted classic method. If the adopted classic method has a precision of polynomials of degree n, then the constructed parametric triangle patches have a precision of parametric polynomials of degree n.

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