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FIRST ORDER ABSOLUTE MOMENT OF MEYER-KÖNIG AND ZELLER OPERATORS AND THEIR APPROXIMATION FOR SOME ABSOLUTELY CONTINUOUS FUNCTIONS

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ABSTRACT. A sharp estimate is given for the first order absolute moment of Meyer-König and Zeller operators M_n. This estimate is then used to prove convergence of approximation of a class of absolutely continuous functions by the operators M_n. The condition considered here is weaker than the condition considered in a previous paper and the rate of convergence we obtain is asymptotically the best possible.

1. Introduction

For a function f defined on $[0,1]$, the Meyer-König and Zeller operators M_n [5] are defined by

$$M_n(f,x)=\sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right) m_{n,k}(x), \qquad 0\le x<1,$$

$$M_n(f,1)=f(1), \qquad m_{n,k}(x)=\binom{n+k}{k} x^k(1-x)^{n+1}. \tag{1}$$

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Let

$$K_{n,x}(t) = \begin{cases} \sum\limits_{k \le nt/(1-t)} m_{n,k}(x), & 0 < t < 1, \\ 1, & t = 1, \\ 0, & t = 0. \end{cases}$$

Then operators M_n have the following Lebesgue-Stieltjes integral representation

$$M_n\,(f,x) = \int_0^1 f(t)\,\mathrm{d}_t K_{n,x}(t). \tag{2}$$

Estimates of the first order absolute moment of the approximation operators play a key role in various investigations of convergence of the approximation operators (for example, cf. [3], [4], [6]–[9], [11]–[13]). In this paper we give a sharp estimate for the first order absolute moment of the operators M_n. Furthermore, by means of this estimate and some analysis techniques we establish a convergence theorem on the approximation of a class of absolutely continuous functions by the operators M_n. The rate of convergence we obtain in this theorem is essentially the best possible.

2. Results and proofs

For the first order absolute moment of Meyer-König and Zeller operators M_n, we have the following result.

Theorem 2.1. *For $x \in (0,1]$, we have*

$$M_n(|t-x|,x) = \frac{\sqrt{2x}(1-x)}{\sqrt{\pi n}} + O\left(\frac{1}{n\sqrt{nx}}\right). \tag{3}$$

Proof. If $x = 1$, (3) is true. Let $0 < x < 1$ and write $r = x/(1-x)$. By the fact that $M_n(t,x) = x$ we have

$$\begin{aligned}
&M_n(|t-x|,x) \\
&= \sum_{k=0}^{[nr]} \left(x - \frac{k}{n+k}\right) m_{n,k}(x) + \sum_{k=[nr]+1}^{\infty} \left(\frac{k}{n+k} - x\right) m_{n,k}(x) \\
&= 2\sum_{k=0}^{[nr]} \left(x - \frac{k}{n+k}\right) m_{n,k}(x) + M_n(t-x,x) \\
&= 2\sum_{k=0}^{[nr]} \binom{n+k}{k} x^{k+1}(1-x)^{n+1} - 2\sum_{k=0}^{[nr]} \frac{k}{n+k} \binom{n+k}{k} x^k (1-x)^{n+1}
\end{aligned}$$

$$
\begin{aligned}
&= 2\sum_{k=0}^{[nr]} \binom{n+k}{k} x^{k+1}(1-x)^{n+1} - 2\sum_{k=0}^{[nr]-1} \binom{n+k}{k} x^{k+1}(1-x)^{n+1} \\
&= 2\binom{n+[nr]}{n} x^{[nr]+1}(1-x)^{n+1}. \qquad (4)
\end{aligned}
$$

Next we estimate

$$2\binom{n+[nr]}{n} x^{[nr]+1}(1-x)^{n+1}.$$

Using Stirling's formula [10], $n! = \sqrt{2\pi n}(n/\mathrm{e})^n \mathrm{e}^{\theta}$, $0 < \theta < 1/12n$, we get

$$2\binom{n+[nr]}{n} = 2\frac{(n+[nr])!}{n![nr]!} = \sqrt{\frac{2}{\pi}}\frac{(n+[nr])^{n+[nr]+1/2}}{n^{n+1/2}[nr]^{[nr]+1/2}}\mathrm{e}^{\theta_1-\theta_2-\theta_3}, \qquad (5)$$

where $0 < \theta_1 < \frac{1}{12(n+[nr])}$, $0 < \theta_2 < \frac{1}{12n}$, $0 < \theta_3 < \frac{1}{12[nr]}$.

Set $c(\theta) = \theta_1 - \theta_2 - \theta_3$, simple calculation derives

$$-\frac{1}{12n} - \frac{1}{12[nr]} < c(\theta) \le 0. \qquad (6)$$

Since $r = x/(1-x)$, by straightforward calculation we have

$$x^{[nr]+1/2}(1-x)^n = \frac{r^{[nr]+1/2}}{(1+r)^{n+[nr]+1/2}}. \qquad (7)$$

Furthermore we find that

$$
\begin{aligned}
&\frac{(n+[nr])^{n+[nr]+1/2}}{n^{n+1/2}[nr]^{[nr]+1/2}}\frac{r^{[nr]+1/2}}{(1+r)^{n+[nr]+1/2}} \\
&= \frac{1}{\sqrt{n}}\left(\frac{nr}{[nr]}\right)^{[nr]+1/2}\left(\frac{n+[nr]}{n+nr}\right)^{n+[nr]+1/2}. \qquad (8)
\end{aligned}
$$

Thus it follows from (5)–(8) that

$$
\begin{aligned}
&2\binom{n+[nr]}{n} x^{[nr]+1}(1-x)^{n+1} \\
&= \sqrt{x}(1-x)2\binom{n+[nr]}{n} x^{[nr]+1/2}(1-x)^n \qquad (9) \\
&= \frac{\sqrt{2x}(1-x)}{\sqrt{\pi n}}\left(\frac{nr}{[nr]}\right)^{[nr]+1/2}\left(\frac{n+[nr]}{n+nr}\right)^{n+[nr]+1/2}\mathrm{e}^{c(\theta)}.
\end{aligned}
$$

Write

$$A(n,r) = \left(\frac{nr}{[nr]}\right)^{[nr]+1/2}\left(\frac{n+[nr]}{n+nr}\right)^{n+[nr]+1/2}, \qquad (10)$$

and

$$nr = [nr] + \nu \qquad (0 \le \nu < 1).$$

Then

$$A(n,r)=\left(1+\frac{\nu}{[nr]}\right)^{[nr]+1/2}\left(1+\frac{\nu}{n+[nr]}\right)^{-(n+[nr]+1/2)}.$$

Thus

$$\begin{aligned}\log A(n,r)&=([nr]+1/2)\log\left(1+\frac{\nu}{[nr]}\right)-(n+[nr]+1/2)\log\left(1+\frac{\nu}{n+[nr]}\right)\\&=([nr]+1/2)\left(\frac{\nu}{[nr]}+O\left(\frac{\nu}{[nr]}\right)^2\right)\\&\quad-(n+[nr]+1/2)\left(\frac{\nu}{n+[nr]}+O\left(\frac{\nu}{n+[nr]}\right)^2\right)\\&=O\left([nr]^{-1}\right),\end{aligned}$$

which means that

$$A(n,r)=1+O\left([nr]^{-1}\right). \tag{11}$$

Hence from (4), (9), (10), (11) and the fact that $\mathrm{e}^{c(\theta)}=1+O(n^{-1}+[nr]^{-1})$, we get

$$\begin{aligned}M_n\left(|t-x|,x\right)&=2\binom{n+[nr]}{n}x^{[nr]+1}(1-x)^{n+1}\\&=\frac{\sqrt{2x}(1-x)}{\sqrt{\pi n}}\left(1+O(n^{-1}+[nr]^{-1})\right)\\&=\frac{\sqrt{2x}(1-x)}{\sqrt{\pi n}}+O\left(\frac{1}{n\sqrt{nx}}\right).\end{aligned}$$

Theorem 2.1 is proved. □

Next we consider approximation of the operators M_n for a class of absolutely continuous functions Φ_{DB} defined by

$$\Phi_{DB}=\Big\{f\mid f(t)-f(0)=\int_0^t h(u)\,\mathrm{d}u,\ \ t\in[0,1],\ \ h \text{ is bounded on } [0,1], \text{ and } h(x+),\ \ h(x-) \text{ exist at } x\in(0,1)\Big\}.$$

The following three quantities are needed in this paper. The readers are referred to the reference [12, p. 244], for their basic properties.

$$\Omega_{x-}(h,\delta_1)=\sup_{t\in[x-\delta_1,\ x]}|h(t)-h(x)|,\qquad \Omega_{x+}(h,\delta_2)=\sup_{t\in[x,\ x+\delta_2]}|h(t)-h(x)|,$$

$$\Omega(x,h,\lambda)=\sup_{t\in[x-x/\lambda,\ x+(1-x)/\lambda]}|h(t)-h(x)|,$$

where h is bounded on $[0,1]$, $x\in[0,1]$ is fixed, $0\le\delta_1\le x$, $0\le\delta_2\le 1-x$, and $\lambda\ge 1$.

We now state the approximation theorem as follows.

Theorem 2.2. *Let $f \in \Phi_{DB}$ and write $\mu = h(x+) - h(x-)$. Then for n sufficiently large we have*

$$\left| M_n(f,x) - f(x) - \mu \frac{\sqrt{x}(1-x)}{\sqrt{2\pi n}} \right| \le \frac{4-2x}{n} \sum_{k=1}^{[\sqrt{n}]} \Omega(x, h_x, k) + \frac{C|\mu|}{n\sqrt{nx}}, \tag{12}$$

where C is a constant independent of n and x, $[\sqrt{n}]$ is the greatest integer not exceeding $\sqrt{n}$ and $h_x(t)$ is defined by

$$h_x(t) = \begin{cases} h(t) - h(x+), & x < t \le 1 \\ 0, & u = x \\ h(t) - h(x-), & 0 \le t < x. \end{cases} \tag{13}$$

In view of the fact that $\frac{1}{\sqrt{n}} \sum_{k=1}^{[\sqrt{n}]} \Omega(x, h_x, k) \to 0$ $(n \to \infty)$, from Theorem 2.2 we get the asymptotic formula

$$M_n(f,x) = f(x) + \frac{\sqrt{x}(1-x)}{\sqrt{2\pi n}} \mu + o(n^{-1/2}),$$

if f satisfies the assumptions of Theorem 2.2. In particular, (12) is true for $f \in DBV[0,1]$ (that is, f is differentiable function whose derivative is of bounded variation, cf. [3]), since the class of functions $DBV[0,1]$ is a subclass of the class Φ_{DB}. We also point out that Abel [1] presented the complete asymptotic expansion for the operators M_n under much stronger conditions.

Moreover, it is of interest to consider some further results. Let f satisfy the assumptions of Theorem 2 and $\Omega(x, h_x, \lambda) = O(1/\lambda)^\alpha$ for some $\alpha > 0$. Then from Theorem 2.2 we get

$$M_n(f,x) = f(x) + \frac{\sqrt{x}(1-x)}{\sqrt{2\pi n}} \mu + \begin{cases} O(n^{-(\alpha+1)/2}), & \text{if } 0 < \alpha < 1 \text{ or } 1 < \alpha < 2 \\ O(\log \sqrt{n}/n), & \text{if } \alpha = 1 \\ O(n^{-3/2}), & \text{if } \alpha \ge 2. \end{cases}$$

Proof of Theorem 2.2

By Bojanic decomposition we have

$$\begin{aligned} h(u) \quad = \quad & \frac{h(x+) + h(x-)}{2} + \frac{h(x+) - h(x-)}{2} \operatorname{sgn}(u-x) \; + \; h_x(u) \\ & + \delta_x(u) \left(h(x) - \frac{h(x+) + h(x-)}{2} \right), \end{aligned} \tag{14}$$

where $\operatorname{sgn}(u)$ is symbolic function, h_x is as defined in (13), and

$$\delta_x(t) = \begin{cases} 1, & t = x \\ 0, & t \ne x. \end{cases}$$

Note that $M_n(t,x) = x$, $\int_x^t \operatorname{sgn}(u-x)\,\mathrm{d}u = |t-x|$, and $\int_x^t \delta_x(u)\,\mathrm{d}u = 0$. From (14) it follows by simple computation that

$$\begin{aligned} f(t) - f(x) &= \int_x^t h(u)\,\mathrm{d}u \\ &= \frac{h(x+)+h(x-)}{2}(t-x) + \frac{h(x+)-h(x-)}{2}|t-x| + \int_x^t h_x(u)\,\mathrm{d}u. \end{aligned}$$

Thus

$$M_n(f,x) - f(x) = \frac{h(x+)-h(x-)}{2} M_n\big(|t-x|, x\big) + M_n\bigg(\int_x^t h_x(u)\,\mathrm{d}u,\ x\bigg). \tag{15}$$

By Lebesgue-Stieltjes integral representation (2) we have

$$\begin{aligned} M_n\bigg(\int_x^t h_x(u)\,\mathrm{d}u,\ x\bigg) &= \int_0^1 \bigg(\int_x^t h_x(u)\,\mathrm{d}u\bigg)\,\mathrm{d}_t K_{n,x}(t) \\ &= L(h,n,x) + Q(h,n,x), \end{aligned} \tag{16}$$

where

$$L(h,n,x) = \int_0^x \bigg(\int_x^t h_x(u)\,\mathrm{d}u\bigg)\,\mathrm{d}_t K_{n,x}(t),$$

$$Q(h,n,x) = \int_x^1 \bigg(\int_x^t h_x(u)\,\mathrm{d}u\bigg)\,\mathrm{d}_t K_{n,x}(t).$$

Integration by parts and note that $K_{n,x}(0) = 0$, $h_x(x) = 0$ we have

$$\begin{aligned} |L(h,n,x)| &= \bigg|\int_0^x K_{n,x}(t) h_x(t)\,\mathrm{d}t\bigg| \\ &\le \int_0^x K_{n,x}(t)\Omega_{x-}(h_x, x-t)\,\mathrm{d}t \\ &= \int_0^{x-x/\sqrt{n}} K_{n,x}(t)\Omega_{x-}(h_x, x-t)\,\mathrm{d}t + \int_{x-x/\sqrt{n}}^x K_{n,x}(t)\Omega_{x-}(h_x, x-t)\,\mathrm{d}t. \end{aligned} \tag{17}$$

By [2, Lemma 2.1] there holds inequality

$$M_n((t-x)^2,x) \le \left(1+\frac{2x}{n-1}\right)\frac{x(1-x)^2}{n+1}.$$

Using this inequality, for $0 \le t < x$ we deduce that

$$\begin{aligned}K_{n,x}(t) &\le \sum_{\frac{k}{n+k}\le t} m_{n,k}(x)\\ &\le \sum_{\frac{k}{n+k}\le t}\left(\frac{k/(n+k)-x}{x-t}\right)^2 m_{n,k}(x)\\ &\le \frac{M_n\left((u-x)^2,x\right)}{(x-t)^2}\\ &\le \frac{1}{(x-t)^2}\left(1+\frac{2x}{n-1}\right)\frac{x(1-x)^2}{n+1}\\ &\le \frac{2x(1-x)^2}{n(x-t)^2}.\end{aligned}$$

Thus by replacement of variable $t = x - x/u$ we have

$$\begin{aligned}\int_0^{x-x/\sqrt{n}} K_{n,x}(t)\Omega_{x-}(h_x,x-t)\,\mathrm{d}t &\le \frac{2x(1-x)^2}{n}\int_0^{x-x/\sqrt{n}}\frac{\Omega_{x-}(h_x,x-t)}{(x-t)^2}\,\mathrm{d}t\\ &= \frac{2(1-x)^2}{n}\int_1^{\sqrt{n}}\Omega_{x-}(h_x,x/u)\,\mathrm{d}u \qquad (18)\\ &\le \frac{2(1-x)^2}{n}\sum_{k=1}^{[\sqrt{n}]}\Omega_{x-}(h_x,x/k).\end{aligned}$$

On the other hand, by inequality $K_{n,x}(t) \le 1$ and the monotonicity of $\Omega_{x-}(h_x,\lambda)$, it follows that

$$\int_{x-x/\sqrt{n}}^{x} K_{n,x}(t)\Omega_{x-}(h_x,x-t)\,\mathrm{d}t \le \frac{x}{\sqrt{n}}\Omega_{x-}\left(h_x,x/\sqrt{n}\right) \le \frac{2x}{n}\sum_{k=1}^{[\sqrt{n}]}\Omega_{x-}(h_x,x/k). \qquad (19)$$

From (19) and (20) and using the basic property $\Omega_{x-}(h_x,\lambda) \le \Omega(x,h_x,x/\lambda)$ (cf. [12, p. 244]) we get

$$|L(h,n,x)| \le \frac{2-2x+2x^2}{n}\sum_{k=1}^{[\sqrt{n}]}\Omega(x,h_x,k). \qquad (20)$$

A similar estimate gives

$$|Q(h,n,x)| \leq \frac{2-2x^2}{n} \sum_{k=1}^{[\sqrt{n}]} \Omega(x,h_x,k). \tag{21}$$

Theorem 2.2 now follows from Eq. (15), (3), (16), (21), and (22).

3. Asymptotic optimality of the estimate in Theorem 2.2

In this section we show that the estimate in Theorem 2.2 is essentially the best possible.

Take function $f(t) = |t-1/2| \in \Phi_{DB}$ at point $x = 1/2 \in (0,1)$. Then $f(1/2) = 0$, $r = x/(1-x) = 1$, $h(u) = \operatorname{sgn}(u-1/2)$, $h_{1/2}(u) \equiv 0$, $h(x+) - h(x-) = 2$, and (12) becomes

$$\left| M_n(|t-1/2|, 1/2) - \frac{1}{2\sqrt{\pi n}} \right| \leq \frac{2\sqrt{2}C}{n^{3/2}}. \tag{22}$$

On the other hand, by straightforward computation and Stirling's formula [10]

$$n! = (2\pi n)^{1/2}(n/\mathrm{e})^n \mathrm{e}^{\theta}, \qquad \left(\frac{1}{12n+1} < \theta < \frac{1}{12n}\right),$$

we get

$$\begin{aligned} M_n\left(|t-1/2|, 1/2\right) &= 2\binom{n+n}{n}\left(\frac{1}{2}\right)^{2n+2} = \frac{(2n)!}{n!n!}\left(\frac{1}{2}\right)^{2n+1} \\ &= \frac{\sqrt{2\pi 2n}(2n/\mathrm{e})^{2n}}{\left(\sqrt{2\pi n}(n/\mathrm{e})^n\right)^2}\left(\frac{1}{2}\right)^{2n+1} \mathrm{e}^{\theta_1 - 2\theta_2} = \frac{1}{2\sqrt{\pi n}}\mathrm{e}^{\theta_1-2\theta_2}, \end{aligned} \tag{23}$$

where

$$\frac{1}{24n+1} < \theta_1 < \frac{1}{24n}, \qquad \frac{1}{12n+1} < \theta_2 < \frac{1}{12n}.$$

Simple computation gives

$$\frac{1}{9n} < \frac{2}{12n+1} - \frac{1}{24n} < 2\theta_2 - \theta_1 < \frac{1}{6n} - \frac{1}{24n+1} < \frac{1}{6n}. \tag{24}$$

Thus, from (24) and (25) we have

$$\left|M_n(|t-1/2|,1/2)-\frac{1}{2\sqrt{\pi n}}\right| = \frac{1}{2\sqrt{\pi n}}\left(1-\mathrm{e}^{\theta_1-2\theta_2}\right) = \frac{1}{2\sqrt{\pi n}}\frac{\mathrm{e}^{2\theta_2-\theta_1}-1}{\mathrm{e}^{2\theta_2-\theta_1}}$$
$$> \frac{1}{2\sqrt{\pi n}}\frac{2\theta_2-\theta_1}{\mathrm{e}^{2\theta_2-\theta_1}} > \frac{1}{2\sqrt{\pi n}}\frac{1/9n}{\mathrm{e}^{1/2}} = \frac{1}{18\sqrt{\pi \mathrm{e}}n^{3/2}}. \tag{25}$$

Eqs. (23) and (26) mean that for $f(t)=|t-1/2|$, the following inequality holds

$$\begin{aligned}\frac{3}{n}\sum_{k=1}^{[\sqrt{n}]}\Omega\left(\frac{1}{2},h_{\frac{1}{2}},k\right)+\frac{1/18\sqrt{\pi\mathrm{e}}}{n\sqrt{n}} &\le \left|M_n\left(f,\frac{1}{2}\right)-f\left(\frac{1}{2}\right)-\frac{1}{2\sqrt{\pi n}}\right| \\ &\le \frac{3}{n}\sum_{k=1}^{[\sqrt{n}]}\Omega\left(\frac{1}{2},h_{\frac{1}{2}},k\right)+\frac{2\sqrt{2}C}{n\sqrt{n}}.\end{aligned} \tag{26}$$

Inequality (27) shows that the estimate (12) in Theorem 2.2 is asymptotically optimal.

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