On graph equivalences preserved under extensions

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Abstract. Let $\mathcal{G}$ be the set of finite graphs whose vertices belong to some fixed countable set, and let $\equiv$ be an equivalence relation on $\mathcal{G}$. By the *strengthening* of $\equiv$ we mean an equivalence relation $\equiv_s$ such that $G \equiv_s H$, where $G, H \in \mathcal{G}$, if for every $F \in \mathcal{G}$, $G \cup F \equiv H \cup F$. The most important case that we study in this paper concerns equivalence relations defined by graph properties. We write $G \equiv^{\Phi} H$, where Φ is a graph property and $G, H \in \mathcal{G}$, if either both G and H have the property Φ, or both do not have it. We characterize the strengthening of the relations $\equiv^{\Phi}$ for several graph properties Φ. For example, if Φ is the property of being a k-connected graph, we find a polynomially verifiable (for k fixed) condition that characterizes the pairs of graphs equivalent with respect to $\equiv_s^{\Phi}$. We obtain similar results when Φ is the property of being k-colorable, edge 2-colorable, hamiltonian, or planar, and when Φ is the property of containing a subgraph isomorphic to a fixed graph H. We also prove several general theorems that provide conditions for $\equiv_s$ to be of some specific form. For example, we find a necessary and sufficient condition for the relation $\equiv_s$ to be the identity. Finally, we make a few observations on the strengthening in a more general case when $\mathcal{G}$ is the set of finite subsets of some countable set.

1 Introduction

Equivalence relations partition their domains into classes of equivalent objects — objects indistinguishable with respect to some characteristic. In the case of domains whose elements can be combined, equivalence relations can be *strengthened*. In this paper, we introduce the concept of strengthening, motivate it, and study it in the case of equivalence relations that arise in the domain of graphs.

To illustrate what we have in mind, let us consider a set X of possible team members. Teams are finite subsets of X. We have some equivalence relation on the set of teams, which groups in its equivalence classes teams of the same value. Thus, given two equivalent teams, say $A, B \subseteq X$, we could use any of them without compromising the quality. But there is more to it. Let us consider a team C, such that A is its sub-team, that is, $A \subseteq C$. Let us also suppose that for one reason or another we are unable to keep all members of A in C. If we need the "functionality" of A in C, we might want to replace A with its equivalent B by forming the team $C' = B \cup (C \setminus A)$. After all, A and B are equivalent. But this is a reasonable solution only if by doing so, we do not change the quality of the overall team, that is, if C and C' are equivalent, too. And, in general, it is not guaranteed.

Let us observe that in our example $C = A \cup (C \setminus (A \cup B))$ and $C' = B \cup (C \setminus (A \cup B)$, that is, they are extensions of A and B, respectively, with the same set, $(C \setminus (A \cup B))$. This suggests that we might call teams A and B *strongly equivalent* (with respect to the original equivalence relation) if for every finite set D, $A \cup D$ and $B \cup D$ are equivalent. Clearly, if A and B are strongly equivalent, then any two teams obtained by extending A and B with the same additional members are equivalent! Thus, the relation of strong equivalence, the "strengthening" of the original one, is precisely what we need when we consider teams not as individual entities but as potential sub-teams in bigger groups.

To the best of our knowledge, the concept of strong equivalence has emerged so far only in the area of logic programming [5, 6, 3, 7]. Researchers argued there that it underlies the notion of a module of a program, and is essential to modular program development. In this paper we study the strengthening of an equivalence relation in the domain of graphs. As a result, we obtain a new class of graph-theoretic problems. Importantly, when applied to specific properties, for instance, to the graph connectivity, the notion of strengthening has interesting practical implication and does give rise to non-trivial arguments and characterizations.

Let us consider the following scenario. In the context of networks, which we typically represent as graphs, the concept of their connectivity is of paramount importance (cf. Colbourn [1]). Let us define graphs G and H to be equivalent if both are k-connected or if neither of them is. With time networks grow and get embedded into bigger networks. The key question is: are the graphs G and H interchangeable, in the sense that the networks obtained by identical extensions of G and H are equivalent with respect to k-connectivity?

For example, neither of the two graphs in Figure 1(a) is 2-connected and so, they are equivalent with respect to 2-connectivity. They are not, however, strongly equivalent with respect 2-connectivity. Indeed, the graphs obtained by extending them with two edges aw and bw, shown in Figure 1(b) are not 2-connectivity equivalent — one of them is 2-connected and the other one is not! On the other hand, one can verify directly from the definition that the graphs shown in Figure 1(c) are strongly equivalent with respect to 2-connectivity. Later in the paper, we provide a characterization that allows us to decide the question of strong equivalence with respect to connectivity.

Our paper is organized as follows. While most of our results concern strengthening of equivalence relations on graphs, we start by introducing the concept of the strengthening of an equivalence relation in a more general setting of the domain of finite subsets of a set. We derive there several basic properties of the notion, which we use later in the paper. In particular, for a class of equivalence relations defined in terms of properties of objects — such equivalence relations are of primary interest in our study — we characterize those relations that are equal to their strengthenings.

The following sections are concerned with equivalence relations on graphs defined in terms of graph properties, a primary subject of interest to us. Narrowing down the focus of our study to graphs allows us to obtain stronger and more interesting results. In particular, we characterize relations whose strengthening is the identity relation, and those whose strengthening defines one large equivalence class, with all other classes being singletons. We apply these general characterizations to obtain descriptions of strong equivalence with respect to several concrete graph-theoretic properties including

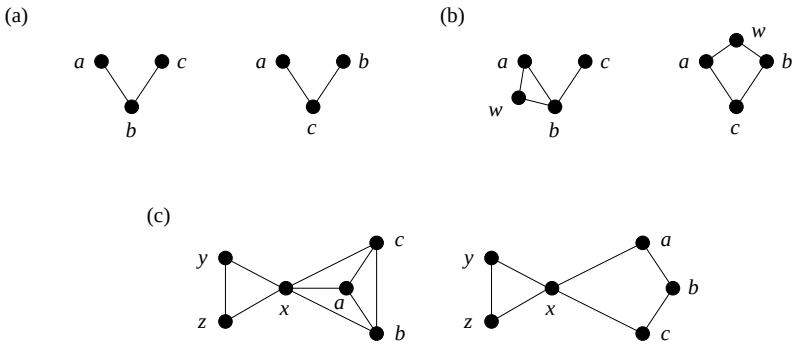

Fig. 1. (a) Graphs that are not strongly equivalent with respect to 2-connectivity; (b) Extensions of the graphs from (a) showing that graphs in (a) are not strongly equivalent with respect to 2-connectivity; and (c) Two graphs that are strongly equivalent with respect to 2-connectivity.

possessing hamiltonian cycles and being planar. Main results of the paper, are contained in the two sections that follow. They concern graph-theoretic properties, which do not fall under the scope of our general results. Specifically, we deal there with vertex and edge colorings, and with k-connectivity. The characterizations we obtain are non-trivial and show that the idea of strengthening gives rise to challenging problems that often (as in the case of strengthening equivalence with respect to k-connectivity) have interesting motivation and are of potential practical interest.

2 The Problem and General Observations

We fix an infinite countable set $\mathcal{E}$ and denote by $\mathcal{G}$ the set of finite subsets of $\mathcal{E}$.

Definition 1. *Let $\equiv$ be an equivalence relation on $\mathcal{G}$. We say that sets $G, H \in \mathcal{G}$ are* strongly equivalent *with respect to $\equiv$, denoted by $G \equiv_s H$, if for every set $F \in \mathcal{G}$, $G \cup F \equiv H \cup F$. We call $\equiv_s$ the* strengthening *of $\equiv$.*

While most of our results concern the case when $\mathcal{E}$ is a set of edges over some infinite countable set of vertices $\mathcal{V}$, in this section we impose no structure on $\mathcal{E}$ and prove several basic general properties of the concept of strong equivalence.

Proposition 1. *Let $\equiv$ be an equivalence relation on sets in $\mathcal{G}$. Then:*

1. *the relation $\equiv_s$ is an equivalence relation*
2. *for every sets $G, H \in \mathcal{G}$, $G \equiv_s H$ implies $G \equiv H$ (that is, $\equiv_s \subseteq \equiv$)*
3. *for every sets $G, H, F \in \mathcal{G}$, $G \equiv_s H$ implies $G \cup F \equiv_s H \cup F$.*

Proof: (1) All three properties of reflexivity, symmetry and transitivity are easy to check. For instance, let us assume that for some three sets $D, G, H \in \mathcal{G}$, $D \equiv_s G$ and $G \equiv_s H$. Let $F \in \mathcal{G}$. By the definition, $D \cup F \equiv G \cup F$ and $G \cup F \equiv H \cup F$. By the transitivity of $\equiv$, $D \cup F \equiv H \cup F$. Since F was an arbitrary element of $\mathcal{G}$, $D \equiv_s H$.

(2) By the definition of $\equiv_s$, for every set $F \in \mathcal{G}$, $G \cup F \equiv H \cup F$. In particular, if $F = \emptyset$, we get that $G \equiv H$.

(3) For every set $F' \in \mathcal{G}$, $F \cup F' \in \mathcal{G}$. Since, $G \equiv_s H$, $(G \cup F) \cup F' = G \cup (F \cup F') \equiv_s H \cup (F \cup F') = (H \cup F) \cup F'$. Thus, the claim follows. □

Proposition 2. *Let $\approx$ and $\equiv$ be equivalence relations on $\mathcal{G}$. Then:*

1. *if $\approx \subseteq \equiv$, then $\approx_s \subseteq \equiv_s$*
2. *if $\approx_s = \equiv_s$, then $(\approx \cap \equiv)_s = \approx_s$.*

Proof: Arguments for each of the assertions are simple. As an example, we prove (2) here. By (1), it suffices to show that $\approx_s \subseteq (\approx \cap \equiv)_s$. Thus, let us consider sets $G, H \in \mathcal{G}$ such that $G \approx_s H$. Let $F \in \mathcal{G}$. Clearly, $G \cup F \approx H \cup F$. Moreover, by the assumption, $G \equiv_s H$. Thus, $G \cup F \equiv H \cup F$, as well. It follows that $G \cup F (\approx \cap \equiv) H \cup F$. As F is arbitrary, $G (\approx \cap \equiv)_s H$ follows. □

Corollary 1. *Let $\equiv$ be an equivalence relation on sets in $\mathcal{G}$. Then, $(\equiv_s)_s = \equiv_s$. Moreover, for every equivalence relation $\approx$ on $\mathcal{G}$ such that $\approx_s = \equiv_s$, $\equiv_s \subseteq \approx$.*

Proof: By Proposition 1(2), $\equiv_s \subseteq \equiv$. Thus, by Proposition 2(1), $(\equiv_s)_s \subseteq \equiv_s$. Conversely, let us consider sets $G, H \in \mathcal{G}$ such that $G \equiv_s H$ and let $F, F' \in \mathcal{G}$. Since $G \equiv_s H$, $G \cup (F \cup F') \equiv H \cup (F \cup F')$. Thus, $(G \cup F) \cup F' \equiv (H \cup F) \cup F'$. As F' is arbitrary, $G \cup F \equiv_s H \cup F$ follows. Consequently, as F is arbitrary, too, $G (\equiv_s)_s H$.

To prove the "moreover" part of the assertion, we note that $\approx_s \subseteq \approx$. Thus, $\equiv_s \subseteq \approx$, as needed. □

Corollary 1 states, in particular, that for every equivalence relation $\equiv$ on $\mathcal{G}$, $\equiv_s$ is the *most precise* among all equivalence relations $\approx$ such that $\approx_s = \equiv_s$.

Most of our results concern equivalence relations defined in terms of functions assigning to sets in $\mathcal{G}$ collections of certain objects. Let $\mathcal{U}$ be a set and let $f : \mathcal{G} \to 2^{\mathcal{U}}$. For sets G and H, we define:

1. $G \cong^f H$ if $f(G) = f(H)$, and
2. $G \equiv^f H$ if $f(G) = f(H) = \emptyset$, or $f(G) \neq \emptyset$ and $f(H) \neq \emptyset$.

Obviously, $\cong^f \subseteq \equiv^f$. Thus, by our earlier results, $\cong^f_s \subseteq \equiv^f_s$, $\equiv^f_s \subseteq \equiv^f$, and $\cong^f_s \subseteq \cong^f$.

Properties of elements of $\mathcal{G}$ (formally, subsets of $\mathcal{G}$) give rise to a special class of equivalence relations of the latter type. Namely, given a property $\Phi \subseteq \mathcal{G}$, we define $\mathcal{U} = \{0\}$ and set $f_\Phi(G) = \{0\}$ if and only if $G \in \Phi$ (otherwise, $f_\Phi(G) = \emptyset$). Clearly, $G \equiv^{f_\Phi} H$ if and only if both G and H have Φ ($G, H \in \Phi$), or if neither G nor H does ($G \notin \Phi$ and $H \notin \Phi$). To simplify the notation, we always write $\equiv^\Phi$ instead of $\equiv^{f_\Phi}$. By $\overline{\Phi}$ we denote the property $\mathcal{G} - \Phi$.

In the remainder of this section we present a general result concerning the relation $\equiv^\Phi$ that does not require any additional structure of subsets of $\mathcal{G}$. It characterizes those properties $\Phi \subseteq \mathcal{G}$, for which $\equiv^\Phi = \equiv^\Phi_s$ (the strengthening does not change the equivalence relation). The remainder of the paper is concerned with the relations $\cong^f$ and $\equiv^f$ (including relations $\equiv^\Phi$) and their strengthenings in the case when $\mathcal{G}$ consists of graphs. In several places, we will consider properties that are monotone. Formally, a property $\Phi \subseteq \mathcal{G}$ is *monotone* if for every $G, H \mathcal{G}$, $G \in \Phi$ implies $H \in \Phi$.

Lemma 1. *Let $\Phi \subseteq \mathcal{G}$ be a property such that $\emptyset \notin \Phi$. Then, $\equiv^{\Phi}=\equiv_s^{\Phi}$ if and only if there is $X \subseteq \mathcal{E}$ such that $\Phi = \{G \in \mathcal{G} \mid G \cap X \neq \emptyset\}$.*

Proof: ($\Leftarrow$) Let us assume that there is $X \subseteq \mathcal{E}$ such that $\Phi = \{G \in \mathcal{G} \mid G \cap X \neq \emptyset\}$. To prove that $\equiv^{\Phi}=\equiv_s^{\Phi}$, it suffices to show that $G \equiv^{\Phi} H$ implies $G \equiv_s^{\Phi} H$ (the converse implication follows by Proposition 1(2)).

Thus, let $G \equiv^{\Phi} H$. First, let us assume that $G, H \in \Phi$. It follows that $G \cap X \neq \emptyset$ and $H \cap X \neq \emptyset$. Consequently, for every set F, $(G \cup F) \cap X \neq \emptyset$ and $(H \cup F) \cap X \neq \emptyset$. Thus, $G \cup F, H \cup F \in \Phi$ and so, $G \cup F \equiv^{\Phi} H \cup F$. It follows that $G \equiv_s^{\Phi} H$.

The only remaining possibility is that $G, H \notin \Phi$. Since $G \cap X = H \cap X = \emptyset$, for every graph F, $(G \cup F) \cap X \neq \emptyset$ if and only if $(H \cup F) \cap X \neq \emptyset$. That is, for every graph F, $G \cup F \equiv^{\Phi} H \cup F$ and so, $G \equiv_s^{\Phi} H$ in this case, as well.

($\Rightarrow$) First, we prove that Φ is monotone. Let $G \subseteq H$ and let $G \in \Phi$. Let us assume that $H \notin \Phi$. It follows that $\emptyset \equiv^{\Phi} H$ and so, by the assumption, $\emptyset \equiv_s^{\Phi} H$. Thus, $G = \emptyset \cup G \equiv^{\Phi} H \cup G = H$. Since $H \notin \Phi$, $G \notin \Phi$, a contradiction. Thus, $H \in \Phi$.

We define $X = \{e \in \mathcal{E} \mid \{e\} \in \Phi\}$. We will show that $G \in \Phi$ if and only if $G \cap X \neq \emptyset$. If $G \cap X \neq \emptyset$, then there is $e \in X$ such that $\{e\} \subseteq G$. Since $\{e\} \in \Phi$, by the monotonicity of Φ it follows that $G \in \Phi$.

Conversely, let us assume that $G \in \Phi$. Let $G' \subseteq G$ be a maximal subset of G such that $G' \notin \Phi$. Such G' exists as $\emptyset \notin \Phi$. Moreover, since $G \in \Phi$, $G' \neq G$. It follows that there is $e \in G \setminus G'$. We have $\emptyset \equiv^{\Phi} G'$ as neither set has property Φ. By the assumption, $\emptyset \equiv_s^{\Phi} G'$. Thus, $\{e\} = \emptyset \cup \{e\} \equiv^{\Phi} G' \cup \{e\}$. By the maximality of G', $G' \cup \{e\} \in \Phi$. Thus, $\{e\} \in \Phi$. Consequently, $e \in X$ and $G \cap X \neq \emptyset$. □

Theorem 1. *Let Φ be a property. Then, $\equiv^{\Phi}=\equiv_s^{\Phi}$ if and only if there is $X \subseteq \mathcal{E}$ such that $\Phi = \{G \in \mathcal{G} \mid G \cap X \neq \emptyset\}$ or $\Phi = \{G \in \mathcal{G} \mid G \subseteq X\}$.*

Proof: ($\Rightarrow$) Let us assume that $\emptyset \notin \Phi$. Then, Lemma 1 implies that there is a set X such that $\Phi = \{G \in \mathcal{G} \mid G \cap X \neq \emptyset\}$. If $\emptyset \in \Phi$, then $\emptyset \notin \overline{\Phi}$. Since $\equiv^{\Phi}=\equiv^{\overline{\Phi}}$, $\equiv_s^{\Phi}=\equiv_s^{\overline{\Phi}}$. Thus, $\equiv^{\overline{\Phi}}=\equiv_s^{\overline{\Phi}}$. By Lemma 1, there is a set Y such that $\overline{\Phi} = \{G \in \mathcal{G} \mid G \cap Y \neq \emptyset\}$. Setting $X = \mathcal{E} \setminus Y$, we obtain that $\Phi = \{G \in \mathcal{G} \mid G \subseteq X\}$.

($\Leftarrow$) If $\Phi = \{G \in \mathcal{G} \mid G \cap X \neq \emptyset\}$, then $\emptyset \notin \Phi$ and the result follows from Lemma 1. Thus, let us assume that $\Phi = \{G \in \mathcal{G} \mid G \subseteq X\}$. We have then that $\overline{\Phi} = \{G \in \mathcal{G} \mid G \cap Y \neq \emptyset\}$, where $Y = \mathcal{E} \setminus X$. Moreover, $\emptyset \notin \overline{\Phi}$. Thus, by Lemma 1, $\equiv_s^{\overline{\Phi}}=\equiv^{\overline{\Phi}}$. By our earlier observations, $\equiv_s^{\Phi}=\equiv^{\Phi}$. □

3 Strengthening of equivalence relations on graphs

From now on we focus on the special case of equivalence relations on graphs. That is, we assume a fixed infinite countable set $\mathcal{V}$ of *vertices* and define $\mathcal{E}$ to be the set of all unordered pairs of two distinct elements from $\mathcal{V}$ (the set of all *edges* on $\mathcal{V}$). Thus, the elements of $\mathcal{G}$ (finite subsets of $\mathcal{E}$) can now be regarded as graphs with the vertex set implicitly determined by the set of edges. For a graph G, we denote by $V(G)$ the set of vertices of G, that is, the set of all endvertices of edges in G. From this convention, it follows that we consider only graphs with no isolated vertices. We understand the

union of graphs as the set-theoretic union of sets. Given a graph G and edges $e \notin G$ and $f \in G$, we often write $G + e$ and $G - f$ for $G \cup \{e\}$ and $G - \{f\}$, respectively. We refer the reader to Diestel [2] for definitions of all graph theoretic concepts not defined in this paper.

Our first result fully characterizes equivalence relations of $\mathcal{G}$ whose strengthening is the identity relation.

Theorem 2. *Let $\equiv$ be an equivalence relation on $\mathcal{G}$. Then, $\equiv_s$ is the identity relation on $\mathcal{G}$ if and only if for every complete graph $K \in \mathcal{G}$ and for every $e \in K$, $K \not\equiv_s K - e$.*

Proof: ($\Rightarrow$) Let K be a complete graph in $\mathcal{G}$ and let $e \in K$. Since $K \neq K - e$, $K \not\equiv_s K - e$, as required.

($\Leftarrow$) Let $G, H \in \mathcal{G}$. Clearly, if $G = H$ then $G \equiv_s H$ (as $\equiv_s$ is reflexive). Conversely, let $G \equiv_s H$. Let us assume that $G \neq H$. Without loss of generality, we may assume that there is an edge $e \in G \setminus H$. Let K be the complete graph on the set of vertices $V(G \cup H)$. Since $G \equiv_s H$, $G \cup (K - e) \equiv_s H \cup (K - e)$ (cf. Proposition 1(3)). Moreover, $G \cup (K - e) = K$ (as $G \subseteq K$ and $e \in G$) and $H \cup (K - e) = K - e$ (as $H \subseteq K$ and $e \notin H$). Consequently, $K \equiv_s K - e$, a contradiction. It follows that $G = H$. □

Remark 1. If we consider finite subsets of an arbitrary infinite countable set $\mathcal{X}$ as objects to compare by equivalence relations, we could prove the following result (by essentially the same method we used in Theorem 2): Let $\equiv$ be an equivalence relation on $\mathcal{P}_{fin}(\mathcal{X})$. Then, $\equiv_s$ is the identity relation on $\mathcal{P}_{fin}(\mathcal{X})$ if and only if for every $S \subseteq \mathcal{X}$ and every $v \in S$, $S \not\equiv_s S - v$. Applying this result to graphs gives us a weaker characterization than the one we obtained, as its condition becomes "for every graph S and every $e \in S$, $S \not\equiv_s S - e$," while our condition is restricted to complete graphs only.

We will now illustrate the applicability of this result. Let $G, H \in \mathcal{G}$ and let us define $G \equiv^{hc} H$ if and only if G and H either both have or both do not have a hamiltonian cycle. Theorem 2 implies that the relation $\equiv_s^{hc}$ is the identity relation. In other words, for every two distinct graphs G and H, there is a graph F such that exactly one of the graphs $G \cup F$ and $H \cup F$ is hamiltonian.

Theorem 3. *Let $G, H \in \mathcal{G}$. Then $G \equiv_s^{hc} H$ if and only if $G = H$.*

Proof: ($\Rightarrow$) Let K be a finite complete graph from $\mathcal{G}$ and let $e \in K$. If $|K| = 1$, then $K = \{e\}$ and $K - e = \emptyset$. Let $e = uv$ and let w be a vertex in $\mathcal{V}$ different from u and v. We define $F = \{uw, vw\}$. Clearly, $K \cup F$ has a hamiltonian cycle and $H \cup F = F$ does not have one. Thus, $K \not\equiv_s^{hc} K - e$. Next, let us assume that $|K| = 3$. Then, K has a hamiltonian cycle and $K - e$ does not. Thus, $K \not\equiv^{hc} K - e$ and, consequently, $K \not\equiv_s^{hc} K - e$.

Finally, let us assume that $|K| \geq 6$ (that is, K is a finite complete graph on at least 4 vertices). Let $v_1, \ldots, v_n$, where $n \geq 4$, be the vertices of K and let $e = v_1 v_2$. We select fresh vertices from $\mathcal{V}$, say $w_3, \ldots, w_{n-1}$. We set $F = \{v_i w_i \mid i = 3, \ldots, n-1\} \cup \{w_i v_{i+1} \mid i = 3, \ldots, n-1\}$. It is clear that $K \cup F$ has a hamiltonian cycle. However, $(K - e) \cup F$ does not have one! Indeed, any such cycle would have to contain all edges

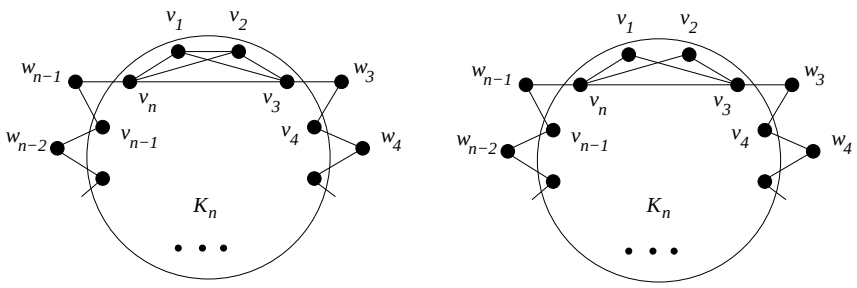

Fig. 2. Graphs $K \cup F$ and $(K - e) \cup F$

in F, and that set cannot be extended to a hamiltonian cycle in $(K - e) \cup F$ (cf. Figure 2). Thus, also in this case $K \not\equiv^{hc} K - e$ and, by Theorem 2, $\equiv_s^{hc}$ is the identity relation.

($\Leftarrow$) This implication is evident. If $G = H$ then, clearly, $G \equiv_s^{hc} H$. □

We note that the strengthening of a related (and more precise) equivalence relation $\cong^{hc}$, where hc is the function that assigns to a graph the set of its hamiltonian cycles, is also the identity relation.

Next, we turn our attention to other equivalence relations on graphs determined by properties of graphs (subsets of $\mathcal{G}$).

We say that a property $\Phi \subseteq \mathcal{G}$ is *strong* if for every $G, H \in \mathcal{G}$, $G \equiv_s^{\Phi} H$ if and only if $G, H \in \Phi$ or $G = H$. In other words, a property Φ is strong, if the strong equivalence with respect to $\equiv^{\Phi}$ does not "break up" the equivalence class Φ of the relation $\equiv^{\Phi}$ (does not distinguish between any graphs with the property Φ) but, in the same time, breaks the other equivalence class into singletons.

We will now characterize properties Φ that are strong.

Theorem 4. *Let Φ be a property of graphs in $\mathcal{G}$ (a subset of $\mathcal{G}$). Then, Φ is strong if and only if Φ is monotone and for every graph $G \notin \Phi$, and every edge $e \in G$, $G \not\equiv_s^{\Phi} G - e$.*

Proof: ($\Leftarrow$) We need to show that for every $G, H \in \mathcal{G}$, $G \equiv_s^{\Phi} H$ if and only if $G, H \in \Phi$ or $G = H$. If $G, H \in \Phi$ then, by the monotonicity of Φ, for every $F \in \mathcal{G}$, $G \cup F \in \Phi$ and $H \cup F \in \Phi$. Thus, $G \cup F \equiv^{\Phi} H \cup F$. Since F is arbitrary, $G \equiv_s^{\Phi} H$. If $G = H$ then $G \equiv_s^{\Phi} H$ is evident.

Conversely, let $G \equiv_s^{\Phi} H$. Then $G \equiv^{\Phi} H$ and so, either both G and H have Φ, or neither G nor H has Φ. In the first case, there is nothing left to prove. Thus, let us assume that $G, H \notin \Phi$.

Let $e \in G \setminus H$. We define $G' = G \cup H$. Since $G \equiv_s^{\Phi} H$, $G' = G \cup H \equiv^{\Phi} H \cup H = H$. Since $H \notin \Phi$, $G' \notin \Phi$. Let G'' be a maximal graph such that $V(G'') = V(G')$, $G' \subseteq G''$ and $G'' \notin \Phi$ (since $G' \notin \Phi$, such a graph exists). We observe that $G'' = G \cup (G'' - e)$ and $G'' - e = H \cup (G'' - e)$. Thus, $G'' \equiv_s^{\Phi} (G'' - e)$ (cf. Proposition 1(3)), a contradiction. It follows that $G \subseteq H$. By symmetry, $H \subseteq G$ and so, $G = H$.

($\Rightarrow$) Let us assume that there is a graph $G \notin \Phi$ such that for every supergraph G' of G, $G' \notin \Phi$. Let H be any *proper* supergraph of G (clearly, G has proper supergraphs). and let F be any graph. Then $G \cup F$ and $H \cup F$ are both supergraphs of G. It follows

that $G \cup F \notin \Phi$ and $H \cup F \notin \Phi$. Consequently, $G \cup F \equiv^{\Phi} H \cup F$. Since F is an arbitrary graph, $G \equiv_s^{\Phi} H$. However, Φ is strong and so the equivalence class of G under $\equiv_s^{\Phi}$ consists of G only (as $G \notin \Phi$). Thus, $G = H$, a contradiction. It follows that for every graph $G \notin \Phi$, there is a supergraph G' of G such that $G' \in \Phi$.

Let us now assume that Φ is not monotone. Then, there are graphs G and H such that $G \subseteq H$, $G \in \Phi$, and $H \notin \Phi$. Let H' be a supergraph of H such that $H' \in \Phi$. It follows that $G \equiv^{\Phi} H'$ and, as Φ is strong, $G \equiv_s^{\Phi} H'$. Thus, $H = G \cup H \equiv^{\Phi} H' \cup H = H'$, a contradiction (as $H \notin \Phi$ and $H' \in \Phi$). It follows that Φ is monotone.

Finally, let $G \notin \Phi$ and $e \in G$. Since Φ is strong, the equivalence class of G under $\equiv_s^{\Phi}$ consists of G only. Consequently, $G \not\equiv_s^{\Phi} G - e$. □

To illustrate the scope of applicability of this result, we will consider now several graph-theoretic properties. We start with the property of non-planarity, that is, the set of all graphs that are not planar.

Theorem 5. *The property of non-planarity is strong.*

Proof: Let Φ denote the property of non-planarity. It is clear that Φ is monotone. Thanks to Theorem 4, to complete the proof it suffices to show that for every graph $G \notin \Phi$ and every edge $e \in G$, $G \not\equiv_s^{\Phi} G - e$.

Thus, let $G \notin \Phi$, that is, let G be a planar graph. Let $e \in G$. We will denote by x and y the endvertices of e. First, let us assume that $G = \{e\}$. Let K be a complete graph on 5 vertices that contains e. Clearly, $\emptyset \cup (K - e)$ is planar and $\{e\} \cup (K - e) = K$ is not. Thus, $G \not\equiv_s^{\Phi} G - e$.

From now on, we will assume that G has at least three vertices. Let G' be a maximal planar supergraph of G such that $V(G) = V(G')$. We will now fix a particular planar embedding of G', and assume that e belongs to the outerface (such an embedding exists). With some abuse of terminology, we will refer also to this embedding as G'. We observe that by the maximality of G', every face in G' is a triangle. Since $G \subseteq G'$, to prove that $G \not\equiv_s^{\Phi} G - e$, it suffices to show that $G' \not\equiv_s^{\Phi} G' - e$ (by Proposition 1(3)).

1. $|V(G')| = 3$. Then G' is a triangle. Let x, y, and z be the vertices of G' and let (as before $e = xy$). Let v and w be two new vertices and $F = \{vx, vy, vz, wx, wy, wz, vw\}$. Then $G' \cup F = K$, where K is a complete graph on 5 vertices. Clearly, K is not planar. On the other hand, $(G' - e) \cup F = K - e$ is planar. Thus, $G' \cup F \not\equiv^{\Phi} (G' - e) \cup F$ and so, $G' \not\equiv_s^{\Phi} G' - e$.

2. $|V(G')| \geq 4$. There are two distinct faces, say F_1 and F_2 in G', sharing the edge e. Both faces are triangles and, without loss of generality, we assume that F_2 is the outerface. Let us assume, as before, that $e = xy$ and let v_1 (respectively, v_2) be the third vertex of the face F_1 (resp. F_2). We note that there is a path from v_1 to v_2 in G' that does not contain x nor y. Indeed, every two edges incident to y and consecutive in the embedding of G' are connected with an edge, as all faces in G' are triangles (cf. Figure 3(a)).

Let v be a new vertex and $F = \{wx, wy, wv_1, wv_2\}$. Then $(G' - e) \cup F$ is planar (cf. Figure 3(b)). On the other hand, the graph $G' \cup F$ contains a subgraph homomorphic to the complete graph on five vertices and so, $G \cup F$ is not planar. Thus, $G' \cup F \not\equiv^{\Phi} (G' - e) \cup F$ and, consequently, $G' \not\equiv_s^{\Phi} G' - e$. □

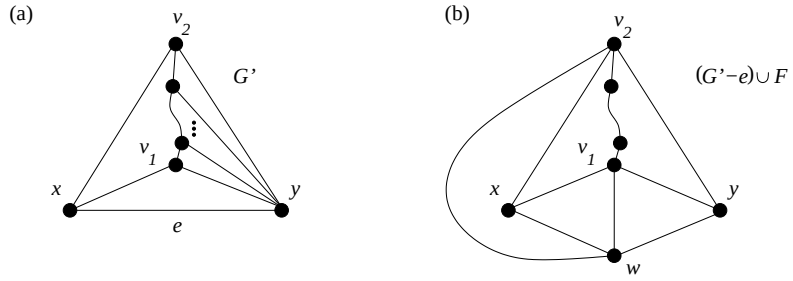

Fig. 3. (a) Graph G'; (b) Graph $G' \cup F$.

Corollary 2. *Let $\equiv^{pl}$ be the equivalence relation such that for every two graphs G and H, $G \equiv^{pl} H$ if both G and H are planar or both G and H are non-planar. Then, for every two graphs G and H, $G \equiv_s^{pl} H$ if and only if both G and H are non-planar or $G = H$.*

Proof: The relation $\equiv^{pl} = \equiv^{\Phi}$, where Φ is the non-planarity property. Since the relation Φ is strong (by Theorem 5), the assertion follows. □

Theorem 4 applies to many graph-theoretic properties concerned with the containment of particular subgraphs. We will present several such properties below.

Theorem 6. *The property Φ_H consisting of all graphs containing a subgraph isomorphic to H is strong in each of the following cases:*

1. *H is a star*
2. *H is a cycle*
3. *H is a 2-connected graph such that for every 2-element cutset $\{x, y\}$, xy is an edge of H*
4. *H is a 3-connected graph*
5. *H is a complete graph*

Proof: In each case the property is monotone. Thus, we only need to show that for every $G \notin \Phi_H$ and every edge $e \in G$, there is a graph F such that $G \cup F \not\equiv^{\Phi_H} (G - e) \cup F$. Below we assume that $e = xy$.

(1) If $H = \{e\}$, for some $e \in \mathcal{E}$, and $G \notin \Phi_H$, then $G = \emptyset$ and so the required property holds vacuously. Thus, let us assume that H consists of $k \geq 2$ edges. Since $G \notin \Phi_H$, $deg_G(x) < k$. Let F be a star with $k - deg_G(x)$ edges all incident to x and with the other end not in G. Clearly, $G \cup F$ contains a star with k edges. On the other hand, $(G - e) \cup F$ does not.

(2) Let H be a cycle with k edges. We define F to be a path with $k - 1$ edges, with endvertices x and y, and with all intermediate vertices not in G. Clearly, $G \cup F$ has a cycle of length k and $(G - e) \cup F$ does not.

(3) Let F' be a graph isomorphic to H, such that $V(F') \cap V(G) = \{x, y\}$ and xy is an edge of F'. Let $F = F' - e$. Clearly, $G \cup F$ contains a subgraph isomorphic to H. Let us assume that $(G-e) \cup F$ contains a subgraph, say H' isomorphic to H. This subgraph

is not contained entirely in $G - e$ (as then G would contain a subgraph isomorphic to H and $G \notin \Phi_H$). Also, H' is not a subgraph of F (F has one fewer edge than H'). Thus, $\{x, y\}$ is a cutset of H' and xy is not an edge of H'. That implies that H has a 2-element cutset whose elements are not joined with an edge, a contradiction.

Parts (4) and (5) of the assertion follow from (1) - (3). Indeed, if a graph H is 3-connected, then it is 2-connected, too. Moreover, it vacuously satisfies the requirement that for every 2-element cutset $\{x, y\}$, xy is an edge of H. Thus, (4) follows. If H is a complete graph, then it is a star (if it consists of only one edge) or a cycle (if it consists of three edges), or is 3-connected. Thus, (5) follows. □

We note that except stars, the 3-edge path is the only tree H such that Φ_H is strong. Indeed, if H is a k-edge tree different from a star and a 3-edge path then we define G to be a complete graph on k vertices and e to be any edge of G. Then, $G \notin \Phi_H$ (as it has only k vertices). However, for every graph F, either both $G \cup F$ and $(G - e) \cup F$ contain H or neither does. Thus, $G \equiv_s^{\Phi_H} G - e$, which implies that Φ_H is not strong.

If H is a 3-edge path then $G \notin \Phi_H$ if and only if every component of G is a star or a triangle. Let $e = xy$ be an edge in G. If e itself is a component of G then let F be a 2-edge path yzu, where z and u are new vertices (not in $V(G)$). If e is an edge of a star centered at x then we define F to be an edge yz, where z is a new vertex. Finally, if e is an edge of a triangle then let F be an edge zu, where z is the third vertex of the triangle and u is a new vertex. In each case $G \cup F$ contains a 3-edge path while $(G - e) \cup F$ does not. Hence $G \not\equiv_s^{\Phi_H} G - e$ which, by Theorem 4, implies that Φ_H is strong when H is the 3-edge path.

Theorem 6 states that for every 3-connected graph H, the property Φ_H is strong. However, the problem of characterizing 1-connected and 2-connected graphs H such that Φ_H is not strong is open.

4 Colorability, Edge Colorability and Connectivity

In the rest of the paper, we discuss strengthening of equivalence relations arising in the context of some well studied graph-theoretic concepts: connectivity, colorability and edge colorability. We start with a simple lemma.

Lemma 2. *Let G and H be graphs. If $V(G) = V(H)$ and the families of vertex sets of components of G and H are not the same then there exists a pair of vertices which are joined by an edge in one of the graphs G or H and are in two different components in the other graph.*

Proof. By our assumptions there exists a component, say G' in G such that $V(G')$ is not the vertex set of any component in H. Let H' be a component in H such that $V(H') \cap V(G') \neq \emptyset$ and let x be any element of $V(H') \cap V(G')$. As $V(G') \neq V(H')$, $V(G') - V(H') \neq \emptyset$ or $V(H') - V(G') \neq \emptyset$. We will consider the former case only because both cases are very similar. Let $y \in V(G') - V(H')$. Since both x and y belong to the component G', there exists a path P in G joining x with y. Clearly, x and y belong to two different components in H so there is an edge in the path P that joins vertices that belong to two different components of H. □

4.1 Colorability

Let k be a positive integer and let the function cl assign to every graph the set of its good k-colorings (to simplify the notation, we drop the reference to k). We will show that $\equiv_s^{cl} = \cong^{cl} = \cong_s^{cl}$.

Theorem 7. *Let k be a positive integer. For every graphs G and H, the following conditions are equivalent:*

(i) $G \equiv_s^{cl} H$,
(ii) $G \cong^{cl} H$,
(iii) $G \cong_s^{cl} H$.

Proof. (i) $\Rightarrow$ (ii). Let $G \equiv_s^{cl} H$. By the inclusion $\equiv_s^{cl} \subseteq \equiv^{cl}$, either both G and H are well k-colorable or both G and H are not well k-colorable. In the latter case the sets of good k-colorings of G and H are empty so they are equal. Thus assume that both G and H are well k-colorable.

Let us suppose $V(G) \neq V(H)$ and assume without loss of generality that there exists a vertex $v \in V(G) - V(H)$. Denote by x a neighbor of v in G. We define a graph F whose vertex set consists of the vertices v, x and some $k-1$ vertices which are not in $V(G) \cup V(H)$. The edges of F join each pair of vertices except v and x. The graph $G \cup F$ contains a complete subgraph K_{k+1} on the vertex set $V(F)$, so $G \cup F$ is not well k-colorable. On the other hand the graph $H \cup F$ is well k-colorable because both H and F are well k-colorable and their only possible common vertex is x. Hence $G \not\equiv_s^{cl} H$, a contradiction.

We have shown that $V(G) = V(H)$. Let us suppose that the sets of good k-colorings of G and H are not equal and assume without loss of generality that there is a good k-coloring $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$ of G, which is not a good k-coloring of H. We define F to be the complete k-partite graph on the set of vertices $V(G)$, whose monochromatic sets of vertices are $C_1, C_2, \ldots, C_k$. Clearly, $\mathcal{C}$ is the only good k-coloring of F. Since $G \cup F = F$, $G \cup F$ has a good k-coloring. On the other hand, $\mathcal{C}$ is not a good k-coloring of H. Thus, it is not a good k-coloring of $H \cup F$, either. We have shown that $G \not\equiv_s^{cl} H$. This contradiction proves that the sets of good k-colorings of G and H are the same.

(ii) $\Rightarrow$ (iii) Let us assume that the sets of good k-colorings of G and H are equal. If both these sets are empty then they remain empty (and so, equal) for the graphs $G \cup F$ and $H \cup F$, for every graph F. If the sets of good k-colorings of G and H are equal but not empty then $V(G) = V(H)$, as good k-colorings of a graph are partitions of the set vertices of this graph. Let F be any graph. If both $G \cup F$ and $H \cup F$ are not well k-colorable then $G \cup F \cong^{cl} H \cup F$. Let now $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$ be a good k-coloring of $G \cup F$. Obviously, $\mathcal{C}' = \{C_1 \cap V(G), C_2 \cap V(G), \ldots, C_k \cap V(G)\}$ is a good k-coloring of G, so by our assumption, $\mathcal{C}'$ is a good k-coloring of H. Thus no block of $\mathcal{C}$ contains an edge of H (as $V(G) = V(H)$). Similarly, no block of $\mathcal{C}$ contains an edge of F because $\mathcal{C}'' = \{C_1 \cap V(F), C_2 \cap V(F), \ldots, C_k \cap V(F)\}$ is a good k-coloring of F. It follows that $\mathcal{C}$ is a good k-coloring of $H \cup F$. Thus the set of good k-colorings of $G \cup F$ is a subset of the set of good k-colorings of $H \cup F$. The converse inclusion can be shown exactly the same way. Consequently, (iii) holds.

(iii) $\Rightarrow$ (i) This implication follows from the obvious inclusion $\cong_s^{cl} \subseteq \equiv_s^{cl}$ (we refer to Proposition 2 and note that $\cong^{cl} \subseteq \equiv^{cl}$). □

Theorem 8. *Let f be the function assigning to every graph the set of its good k-colorings. If $k \geq 3$ then the problem of deciding if $G \not\equiv_s^{cl} H$ is NP-complete.*

Proof. By Theorem 7, to demonstrate that $G \not\equiv_s^{cl} H$, it suffices to show a k-coloring of one of the graphs G or H which is not a k-coloring of the other graph. Hence the problem is in NP.

We will now reduce the NP-complete problem of existence of a k-coloring of a graph (cf. Garey and Johnson [4]) to our problem. Let G' be a graph. We define G to be the graph obtained from G' by adding an edge xy, where x and y are two new vertices. We define H to be the graph obtained from G by adding an edge zx, where z is some vertex of G'. We will prove that G' is k-colorable if and only if $G \not\equiv_s^{cl} H$.

Let us suppose first that G' is k-colorable. Then there exists a k-coloring of G such that both vertices x and z belong to the same block of the k-coloring. This k-coloring of G is not a k-coloring of H because xz is an edge in H. Thus $G \not\cong^{cl} H$ and, by Theorem 7, $G \not\equiv_s^{cl} H$. Conversely, if G' is not k-colorable then neither is G nor H. Hence, the sets of k-colorings of both G and H are empty and, consequently, equal. By Theorem 7 again $G \equiv_s^{cl} H$. □

Remark 2. For $k = 1$, that is, when the function cl assigns to a graph the set of its good 1-colorings of a graph, all four equivalence relations $\equiv^{cl}, \cong^{cl}, \equiv_s^{cl}$ and $\cong_s^{cl}$ coincide and for every pair of graphs G and H, $G \equiv^{cl} H$ if and only if $G = H = \emptyset$ or $G \neq \emptyset \neq H$. □

For $k = 2$ the problem if $G \not\equiv_s^{cl} H$ is solvable in polynomial time. It is a consequence of the following fact.

Proposition 3. *Let cl be the function assigning to every graph the set of its good 2-colorings. For any two graphs G and H, $G \equiv_s^{cl} H$ if and only if either none of the graphs G and H is bipartite or*

1. *both G and H are bipartite*
2. *$V(G) = V(H)$*
3. *G and H have the same families of vertex sets of connected components, and*
4. *connected components with the same vertex sets in G and H have the same bipartitions.*

Proof. ($\Rightarrow$) Let $G \equiv_s^{cl} H$ and assume that at least one of the graphs G or H is bipartite. Otherwise the necessity holds. As $\equiv_s^{cl} \subseteq \equiv^{cl}$, both G and H are bipartite. Hence the sets of good 2-colorings of G and H are nonempty and they are equal by Theorem 7. Since good k-colorings in a graph are partitions of the vertex set of this graph, $V(G) = V(H)$.

Let us suppose the families of vertex sets of components of G and H are not the same. Then, by Lemma 2, there exists a pair of vertices that are joined by an edge in one of the graphs G or H and are in different components in the other graph. Without loss of generality we can assume that there are vertices, say x and y, that are joined by an edge in G but belong to two different components in H. Let us denote by H_x (respectively,

H_y) the two components in H that contain x (respectively, y) and by V_x (respectively, V_y) the monochromatic class of H_x (respectively, H_y) that contains x (respectively, y). Let F be the complete bipartite graph on the set of vertices $V(H_x) \cup V(H_y)$ whose monochromatic classes are $V_x \cup V_y$ and the other one $(V(H_x) \cup V(H_y)) - (V_x \cup V_y)$. The graph $H \cup F$ is bipartite while $G \cup F$ is not because it contains the graph $F + xy$ which has an odd cycle. By the definition of the relation $\equiv_s^{cl}$, $G \not\equiv_s^{cl} H$, a contradiction. Hence for each component in G, there is a component in H with the same vertex set. By symmetry, G and H have the same families of vertex sets of their connected components.

Let us suppose now that for some two components G' of G and H' of H with the same vertex sets, the bipartitions of G' and H' are not the same. Then there exists a pair of vertices u and v such that they both are in the same monochromatic class in G' but in the different monochromatic class in H'. The graph $(G \cup H') + uv$ is not bipartite because there is a path of an even length joining u and v in G so $(G \cup H') + uv$ contains an odd cycle. On the other hand the graph $(H \cup H') + uv = H + uv$ is bipartite. Hence $G \not\equiv_s^{cl} H$, a contradiction. This completes the proof of necessity.

($\Leftarrow$) If both G and H are not bipartite then for every graph F, both $G \cup F$ and $H \cup F$ are not bipartite so $G \equiv_s^{cl} H$. Let now both G and H be bipartite. Let us denote by $\mathcal{C}$ a good 2-coloring of G and suppose $\mathcal{C}$ is not a good 2-coloring of H. Then there exists an edge, say e, in H whose both ends are contained in the same block of $\mathcal{C}$. Let H' be the component of H that contains this edge. By our assumptions, there is a component G' of G that has the same vertex set and the same bipartition as H'. Thus the edge e has its ends in two different blocks of this bipartition of G' so in two different blocks of $\mathcal{C}$ as well. This contradiction proves that every good 2-coloring of G is a good 2-coloring of H. In a very similar way one can prove that every good 2-coloring of H is a good 2-coloring of G. By Theorem 7, we conclude that $G \equiv_s^{cl} H$. □

4.2 Edge 2-colorability

We will now consider the property of edge 2-colorability. We note that a graph is *edge 2-colorable* if and only if each of its connected components is a path or a cycle of an even length We denote by $\equiv^{e2c}$ the equivalence relation in which two graphs are equivalent if and only if both are edge 2-colorable or neither of the two is.

The following theorem characterizes the relation $\equiv_s^{e2c}$.

Theorem 9. *Let $G, H \in \mathcal{E}$. Then $G \equiv_s^{e2c} H$ if and only if at least one of the following conditions holds*

1. *Both G and H are not edge 2-colorable (each contains an odd cycle or a vertex of degree at least 3)*
2. *Both G and H are edge 2-colorable (no odd cycles and maximum degree at most 2), $V(G) = V(H)$, and for every even (odd) path component in G there is an even (odd) path component in H with the same endpoints.*

Proof. ($\Rightarrow$) If both G and H are not edge 2-colorable, there is nothing left to prove. Since $G \equiv_s^{e2c} H$, then $G \equiv^{e2c} H$ and, consequently, both G and H are edge 2-colorable. Let us suppose that $V(G) \neq V(H)$, say $V(H) - V(G) \neq \emptyset$. Let $u \in$

$V(H) - V(G)$. We denote by v and w some new vertices (occurring neither in G nor in H). We define $F = \{vu, wu\}$. Clearly, $G \cup F$ is edge 2-colorable, while $H \cup F$ is not. Thus, $G \not\equiv_s^{e2c} H$, a contradiction, so $V(G) = V(H)$.

Suppose now there is a vertex u of G such that $deg_G(u) = 1$ and $deg_H(u) = 2$. Let v be a new vertex (occurring neither in G nor in H). We define $F = \{vu\}$. Obviously, as before, $G \cup F$ is edge 2-colorable, while $H \cup F$ is not. Hence, $G \not\equiv_s^{e2c} H$, a contradiction. By the symmetry argument it follows that for every vertex u, $deg_G(u) = deg_H(u)$.

Let now P be a path in G with endpoints a and b. By the property proved above, $deg_H(a) = deg_H(b) = 1$. Let us assume that a and b are the endpoints of two different paths in H. We select a new vertex, say u. If P has odd length, we define $F = \{au, ub\}$. Otherwise, we define $F = \{ab\}$. Clearly, $G \cup F$ contains an odd cycle and so, it is not edge 2-colorable. On the other hand, $H \cup F$ does not contain any odd cycles and so, it is edge 2-colorable, a contradiction.

Thus, a and b are the endpoints of the same path in H, say P'. It remains to prove that the length of P' is of the same parity as the length of P. If $G + ab$ is edge 2-colorable, then the cycle $P + ab$ has even length. Thus, the cycle $P' + ab$ is also even. Thus, both P and P' are both of odd length. If $G + ab$ is not edge 2-colorable, then $P + ab$ is an odd cycle. It follows that $P' + ab$ is an odd cycle, too and, consequently, P and P' are both of even length.

($\Leftarrow$) Let F be any set of edges and let us assume that $G \cup F$ is edge 2-colorable. It follows that G is edge 2-colorable and so, H is edge 2-colorable, too. Moreover, no vertex in $G \cup F$ has degree 3 or more. By our assumptions, the same holds for $H \cup F$ because the degrees of vertices in G and H are the same. Let us consider any cycle C in $H \cup F$. If $C \cap F = \emptyset$, then $C \subseteq H$. Consequently, C is even. Thus, let $F' = C \cap F$. By our assumptions, adding F' to G results in exactly one new cycle in G, say C'. Moreover, the parity of the lengths of C' and C is the same. Since $G \cup F$ is edge 2-colorable and contains C', C' is even. Thus, C is even, too. It follows that $H \cup F$ is edge 2-colorable. By symmetry, for every graph F, $G \cup F$ is edge 2-colorable if and only if $H \cup F$ is edge 2-colorable. □

4.3 Connectivity

By a *cutset* in a connected graph we mean a set of vertices in this graph whose deletion makes this graph disconnected. A set C of vertices in a disconnected graph G is a *cutset* of G if $C = \emptyset$ or, for some component G' of G, $C \cap V(G')$ is a cutset of G'. Clearly, $C \neq \emptyset$ is a cutset of G if and only if it separates some pair of vertices in G. Let us observe that the only graphs without any cutsets are the complete graphs.

Let cs be a function that assigns to every graph the set of its cutsets of cardinality smaller than k, where $k \geq 1$ (as in the case of colorability, to simplify the notation, we drop the reference to k). Thus, $G \equiv^{cs} H$, if either both graphs G and H have a cutset of cardinality less than k or both do not have such a cutset. We shall characterize now the relation $\equiv_s^{cs}$.

Lemma 3. *If* $G \equiv_s^{cs} H$ *then* $V(G) = V(H)$.

Proof. Let us suppose that $V(G) \neq V(H)$. We can assume without loss of generality that there exists a vertex $x \in V(H) - V(G)$.

We will first assume that $k = 1$. Let z be any vertex in H different from x and let y be a new vertex (not in H nor G). We define $F = \{zu \mid u \in V(H) - \{x, z\}\} \cup \{xy\}$. It follows that $H \cup F$ is connected (that is, has no cutsets of size 0) and $G \cup F$ is not connected (the edge xy is separated from the rest of the graph). Thus, $G \not\equiv_s^{cs} H$, a contradiction.

Thus, from now on, we assume that $k \geq 2$. Let K be a set of k vertices which are not in $V(G) \cup V(H)$ and let $\ell = \max(k - \deg_H(x), 1)$. Since x is not an isolated vertex in H and $k \geq 2$, $\ell < k$. We define F to be the graph obtained from the complete graph on $(V(H) \cup K) - \{x\}$ by adding the vertex x and edges joining x with all vertices of some ℓ-element subset L of K. The graph $H \cup F$ can be obtained from F by adding the edges incident in H with x. As $\deg_{H \cup F}(x) = \deg_H(x) + \ell \geq k - \ell + \ell = k$, the graph $H \cup F$ does not have a cutset of cardinality smaller than k. On the other hand, the set L is a cutset in $G \cup F$. Indeed, L is a cutset of the component of $G \cup F$ containing $K \cup \{x\}$ as it separates x from the rest of the graph. Since $|L| < k$, $G \cup F$ has a cutset of cardinality smaller than k. Thus, $G \not\equiv_s^{cs} H$, a contradiction, and so $V(G) = V(H)$ follows. □

Theorem 10. *Let G and H be graphs. Then, $G \equiv_s^{cs} H$ if and only if $V(G) = V(H)$ and for every set $C \subseteq V(G)$ such that $|C| < k$, the families of vertex sets of components of $G - C$ and $H - C$ are the same.*

Proof. ($\Leftarrow$) To show sufficiency assume that $G \not\equiv_s^{cs} H$. Then, for some graph F, $G \cup F \not\equiv^{cs} H \cup F$. We can assume without loss of generality that $G \cup F$ has a cutset C of cardinality smaller than k and $H \cup F$ does not have such a cutset. As $(G \cup F) - C$ is disconnected, there are vertices x and y which belong to two different components of $(G \cup F) - C$. On the other hand $(H \cup F) - C$ is connected so there exists a path, say P, joining x and y in $(H \cup F) - C$. Let e be any edge in P which does not belong to F. Then, clearly, e is an edge of $H - C$. We denote by H' the component of $H - C$ which contains the edge e. Since the families of vertex sets of components of $G - C$ and $H - C$ are the same, there is a component of G that contains the edge e. Consequently there exists a path, say P_e in $G - C$ joining the ends of the edge e. Let us replace in the path P every edge e which is not in F by the path P_e. The resulting graph is a connected subgraph of $(G \cup F) - C$ containing the vertices x and y. We have got a contradiction with the definition of x and y. Thus our initial assumption that $G \not\equiv_s^{cs} H$ was false, so $G \equiv_s^{cs} H$.

($\Rightarrow$) We pass on to the proof of necessity. By Lemma 3, $V(G) = V(H)$.

Let $C \subseteq V(G)$, $|C| < k$. Let us suppose the families of vertex sets of components of $G - C$ and $H - C$ are not the same. By Lemma 2, there exists a pair of vertices which are joined by an edge in one of the graphs G or H and are in two different components in the other graph. Without loss of generality we assume that there is an edge e in H whose ends belong to two different components in G. Let G' be one of these components. We denote by L a set of $\ell = k - 1 - |C|$ vertices which are not in $V(G) \cup V(H)$. We define F to be the graph on the set of vertices $V(G) \cup L$ in which every pair of vertices in $V(G')$ is joined by an edge, every pair of vertices in $V(G) - V(G')$ is joined by an edge and every vertex in $C \cup L$ is joined by an edge with every other vertex of F. Clearly, $C' = C \cup L$ is a cutset in $G \cup F = F$ of cardinality

$|C'| = |C| + k - 1 - |C| = k - 1$. Let us consider any cutset C'' in $H \cup F$. Since $V(H \cup F) = V(H) \cup L = V(F) = V(G) \cup L = V(G \cup F)$ and every vertex in C' is joined by an edge with every other vertex of $H \cup F$, $C'' \supseteq C'$. Let us observe that $(H \cup F) - C' = (H \cup F) - C \supseteq (F - C) + e$. The last graph is a connected spanning subgraph of $(H \cup F) - C'$, so C' is not a cutset of $H \cup F$. We have shown that every cutset in $H \cup F$ has at least k vertices which shows that $G \not\equiv_s^{cs} H$. This contradiction completes the proof. □

Remark 3. It follows from Theorem 10 that for every fixed k, the problem to decide if $G \equiv_s^{cs} H$ is polynomial time solvable, where cs is the function assigning to every graph the set of its cutsets of cardinality smaller than k. It is an open question what the complexity status of this problem is when k is a part of the instance. □

Next, we will show that $\equiv_s^{cs} = \cong_s^{cs}$. That is, $\cong^{cs}$ despite being more precise than $\equiv^{cs}$ has the same strangthening.

Theorem 11. *$G \equiv_s^{cs} H$ if and only if $G \cong_s^{cs} H$.*

Proof. ($\Leftarrow$) This implication follows from a generally true inclusion $\cong_s^f \subseteq \equiv_s^f$.

($\Rightarrow$) Let G and H be graphs such that $G \equiv_s^{cs} H$. We assume that there is a graph F such that $G \cup F \not\cong^{cs} H \cup F$. Since $G \equiv_s^{cs} H$, by Proposition 1(3), $G \cup F \equiv_s^{cs} H \cup F$. By Lemma 3, $V(G \cup F) = V(H \cup F)$. As $G \cup F \not\cong^{cs} H \cup F$, there exists a cutset C in $G \cup F$, $|C| < k$, which is not a cutset in $H \cup F$. Let x and y be a pair of vertices such that C separates them in $G \cup F$. We denote by G' the component of $(G \cup F) - C$ that contains x. Clearly C does not separate x and y in $H \cup F$, so there is a path in $(H \cup F) - C$ that joins the vertices x and y. Consequently there exists an edge, say e, in $(H \cup F) - C$ with one vertex in $V(G')$ and the other one in $V((H \cup F) - C) - V(G') = V((G \cup F) - C) - V(G')$.

Let L be a set of cardinality $k-1-|C|$ of vertices not in $V(G \cup F)$. We define K to be the graph on the set of vertices $V(G \cup F) \cup L$ in which every pair of vertices in $V(G')$ is joined by an edge, every pair of vertices in $V((G \cup F) - C) - V(G')$ is joined by an edge and every vertex in $C' = C \cup L$ is joined by an edge with every other vertex of K. Clearly, C' is a cutset in $G \cup F \cup K = K$ of cardinality $|C'| = |C| + k - 1 - |C| = k - 1$. Since $G \equiv_s^{cs} H$, the graph $H \cup F \cup K$ has a cutset C'' of cardinality smaller than k. We observe that $C'' \supseteq C'$ because $V(H \cup F \cup K) = V(K)$ and every vertex in C' is joined by an edge with every other vertex in $V(H \cup F \cup K)$. As $|C''| < k$, $C'' = C'$. By the definition of K, $V((H \cup F \cup K) - C') = V(K - C') = V((G \cup F) - C)$ and the graph $(H \cup F \cup K) - C'$ contains complete graphs on the sets of vertices $V(G')$ and $V((G \cup F) - C) - V(G')$ as subgraphs. The graph $(H \cup F \cup K) - C'$ contains the edge e whose one end is in one of the complete subgraphs mentioned above and the other one in the other complete subgraph. Therefore the graph $(H \cup F \cup K) - C'$ is connected so $C'' = C'$ is not a cutset in $H \cup F \cup K$, a contradiction.

We have shown that $G \cup F \cong^{cs} H \cup F$ for every graph F, so $G \cong_s^{cs} H$. □

Remark 4. Unlike in the case of k-coloring, $\cong^{cs} \neq \cong_s^{cs}$. A simple example for $k = 2$ is shown in Figure . 4.

Fig. 4. The set $\{c\}$ is the only one-element cutset of G and of H. Thus, $G \cong^{cs} H$. However, components of $G - c$ and $H - c$ are different and so, $G \not\cong_s^{cs} H$. □

For a given graph G it would be interesting to find a smallest (with respect to the number of edges) subgraph G' of G such that $G' \equiv_s^{cs} G$. We observe, however, that even for $k = 2$, it is a difficult problem. Indeed, let us consider the problem of deciding if for a given graph G and an integer m there exists a subgraph G' of G such that $G \equiv_s^{cs} G'$ and G' has at most m edges. One can easily verify that the problem is in the class NP (cf. Remark 3). Moreover it is NP-complete because it follows from Theorem 10 that for a 2-connected graph G and $m = |V(G)|$ the problem asks for the existence of a hamiltonian cycle in G.

Let Φ be the set of all graphs with a cutset of cardinality smaller than k. It is obvious that $\equiv^{\Phi} = \equiv^{cs}$. Let us denote by Ψ the set of graphs that are not k-connected. One can easily observe that a graph $G \in \Psi$ if and only if G has a cutset of cardinality smaller than k or $|V(G)| \leq k$. As the two relations are closely related, it is natural to ask if $\equiv_s^{\Phi} = \equiv_s^{\Psi}$. We will answer this question positively.

Theorem 12. *$G \equiv_s^{\Phi} H$ if and only if $G \equiv_s^{\Psi} H$.*

Proof. ($\Rightarrow$) Let us suppose $G \equiv_s^{\Phi} H$ but $G \not\equiv_s^{\Psi} H$. Then, without loss of generality, there exists a graph F such that $G \cup F \in \Psi$ and $H \cup F \notin \Psi$. In this case, $H \cup F$ has no cutsets of cardinality smaller than k and $|V(H \cup F)| > k$. Since $G \equiv_s^{\Phi} H$, $G \cup F$ has no cutsets of cardinality smaller than k. Thus, $|V(G \cup F)| \leq k$ because $G \cup F \in \Psi$. Hence $|V(G \cup F)| < |V(H \cup F)|$, a contradiction because, by Lemma 3, $V(G) = V(H)$, so $V(G \cup F) = V(H \cup F)$, as well.

($\Leftarrow$) Let us suppose now that $G \equiv_s^{\Psi} H$ but $G \not\equiv_s^{\Phi} H$. Then, without loss of generality, there exists a graph F such that $G \cup F$ has a cutset of cardinality smaller than k and $H \cup F$ does not have such a cutset. It follows that $G \cup F \in \Psi$. Since $G \equiv_s^{\Psi} H$, $H \cup F \in \Psi$. Thus, $|V(H \cup F)| \leq k$ and, consequently, $H \cup F$ is a complete graph on at most k vertices. As $G \equiv_s^{\Psi} H$, $G \cup H \cup F \equiv^{\Psi} H \cup H \cup F = H \cup F$, so $G \cup H \cup F \in \Psi$.

Let us suppose first that $G \cup H \cup F$ has a cutest C of cardinality smaller that k. Let G' be one of the components of $(G \cup H \cup F) - C$ and let $G'' = (G \cup H \cup F) - C - G'$. The complete graph $H \cup F$ has common vertices with at most one of the graphs G' and G'', say with G''. Let H' be a complete graph on $k+1$ vertices that contains $H \cup F$ and has no common vertices with G'. Now, $G \cup H \cup F \cup H' \in \Psi$ because C is its cutset, while $H \cup H \cup F \cup H' = H' \notin \Psi$ because it is a complete graph on $k + 1$ vertices. Hence $G \not\equiv_s^{\Psi} H$, a contradiction.

We have proved that $G \cup H \cup F$ has no cutsets of cardinality smaller that k so, as $G \cup H \cup F \in \Psi$, $|V(G \cup H \cup F)| \leq k$. Consequently, $|V(G \cup H)| \leq k$. Since $G \not\equiv_s^{\Phi} H$, $G \neq H$ have different edge sets. We can assume without loss of generality that there is an edge, say $e = xy$ in H, which is not an edge in G. Let K be a graph obtained

from the complete graph on a $(k+1)$-element set of vertices containing $V(G \cup H)$ by deleting the edge e. Clearly, $G \cup K = K$ but $H \cup K$ is the complete graph on $k+1$ vertices. The former graph has a cutset $V(K) - \{x, y\}$ of cardinality $k-1$ and the latter graph has no cutsets. Hence $G \cup K \in \Psi$ and $H \cup K \notin \Psi$ so $G \not\equiv_s^{\Psi} H$. This contradiction proves that $G \equiv_s^{\Phi} H$. □

5 Open problems and further research directions

We do not know of any past research concerning the strengthening of equivalence relations on graphs. Nevertheless, it seems to us that this is a natural concept worth further investigations. In this paper, we focused on the strengthening of the equivalence relations that are determined by graph properties. For many properties Φ we studied, the relations $\equiv_s^{\Phi}$ turned out to have a very simple structure (for instance, they broke both equivalence classes of $\equiv^{\Phi}$, Φ and $\overline{\Phi}$, into singletons, that is, were identities; or kept Φ as an equivalence class and broke the other one into singletons). In several cases, however, (for the properties Φ of being k-connected, k-colorable, and edge 2-colorable), the structure of the relations $\equiv_s^{\Phi}$ turned out to be more complex and so more interesting, too. Therefore, a promising research direction could be to identify and study additional natural graph properties Φ, for which the relations $\equiv_s^{\Phi}$ have a nontrivial structure. Establishing characterizations of the relations $\equiv_s^{\Phi}$, and determining the complexity of deciding whether for two given graphs G and H, $G \equiv_s^{\Phi} H$ holds, are particularly interesting and important. Given the results of our paper, it seems that the properties of being edge k-colorable and edge k-connected are natural candidates for this kind of investigations. In the former case we were only able to find a characterization of the relation $\equiv_s^{\Phi}$, when Φ is the property of being edge 2-colorable. The theorem we proved in this case suggests that for an arbitrary k the structure of the relation $\equiv_s^{\Phi}$ may be quite complicated, which makes the problem a challenge. In the latter case, we feel there may be strong similarities with the strengthening of the property of k-connectivity but do not have any specific results.

There are a few open problems directly related to the results of this paper. One of them is to establish the computational complexity of the problem to decide if $G \equiv_s^{\Phi} H$ (given graphs G and H, and an integer k), when Φ is the property of being k-connected. For a fixed k, it follows from our Theorem 10 that the problem is solvable in polynomial time. The question is open however, when k is a part of the instance.

Another problem concerns the property Φ_H of containing a subgraph isomorphic to H. It was shown in Theorem 6 that the relation $\equiv_s^{\Phi_H}$ has some very simple structure for many graphs H. The question arises what is the structure of $\equiv_s^{\Phi_H}$ for all other graphs H.

There are also several natural general questions concerning the concept of strengthening of an equivalence relation in graphs. For example it would be interesting to find a general condition for the function f that guarantees that the relations $\equiv_s^f$ and $\cong_s^f$ are equal.

Another general problem is to establish conditions that ensure that there exists a weakest equivalence relation $\equiv'$ such that $\equiv'_s$ is the same as $\equiv_s$ and, whenever it is so, to find this $\equiv'$. In some cases, the problem is easy. For example, the relations $\equiv_s^{cl}$ and

$\cong_s^{cl}$ studied in Subsection 4.1 are equal (see Theorem 7) and the relation $\equiv^{cl}$ is strictly weaker than $\cong^{cl}$. As $\equiv^{cl}$ has only two equivalence classes and the strengthening of the total relation (the only possible weakening of $\equiv^{cl}$) is also the total equivalence relation, $\equiv^{cl}$ is the weakest relation $\equiv'$ such that $\equiv'_s$ and $\cong_s^{cl}$ are equal. This observation does not generalize to other properties. For instance, the relation $\equiv^{cs}$ is not the weakest relation $\equiv'$ such that $\equiv'_s$ and $\cong_s^{cs}$ are the same. Indeed, the strengthening of $\equiv^{\Psi}$ (where Ψ is the property considered in Theorem 12) coincides with $\cong_s^{cs}$, yet $\equiv^{cs}$ is incomparable with $\equiv^{\Psi}$ (and so, in this case, there is no weakest relation of the desired property). Thus, in general, the two problems mentioned above seem to be nontrivial.

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