A Parameterization Approach for Quadrilateral Mesh

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Abstract Parameterization of a 3D mesh is a fundamental problem in various applications of meshes. The approaches are widely used for parameterization because of its good properties, but they are almost based on triangle mesh. In this paper, we present a parameterization approach for the quadrilateral open mesh with complex topology. Since mesh simplification and weighted discrete mapping are adapted, the parameterization approach reduces the computation of the mapping process and can better reflect the shape of the corresponding 3D mesh.

Key words Parameterization, Mesh simplification, Gaussian curvature, Optimization.

1 Introduction

Parameterization is an important problem in Computer Graphics. A parameterization of a polygonal mesh in 3D space can be viewed as a one-to-one mapping from the given mesh to a suitable domain. Parameterization has many applications in various fields, including texture mapping [1,2], scattered data and surface fitting[3-5], multi-resolution modeling [6,7], remeshing [8,9] and morphing[10], and so on.

Due to its primary importance for subsequent mesh manipulation, the subject of the mesh parameterization has been researched for a number of years. The major paradigms in mesh parameterization are energy functional minimization and the convex combination approach. Several approaches have been developed to define and minimize an energy functional that measures distortion in the embedded mesh. Maillot proposed a method to minimize a norm of the Green–Lagrange deformation tensor based on elasticity theory [11]. The harmonic embedding used by Eck minimizes the metric dispersion instead of elasticity[6]. Lévy proposed an energy functional minimization method based on orthogonality and homogeneous spacing[12]. Non-deformation criterion is introduced in[13] with extrapolating capabilities.

The convex combination approach is an extension of the barycentric mapping approach proposed by Tutte[14]. This approach obtains parameterization by fixing the boundary vertices of a 3D mesh onto a 2D convex polygon and solving a linear system to determine the 2D embedded positions of the interior vertices. Floater[15] proposed shape-preserving parameterization, where the coefficients are determined by using conformal mapping and barycentric coordinates. The harmonic embedding [6] is also a special case of this approach, except that the coefficients may be negative.

The previous works focus on the triangular mesh parameterization. Because of the good properties of the quadrilateral mesh, it is used more and more, which makes the parameterization of quadrilateral meshes become a critical problem. However, the parameterization approaches of triangle meshes are not suitable for that of quadrilateral meshes due to different connectivity structure.

In this paper, we propose a parameterization approach for a quadrilateral mesh. We firstly map the simplified mesh whose interior vertices with low Gaussian curvature are deleted on the plane, and then embed the interior vertices on the parameterization domain. The proposed approach has the following advantages: (1) The method provably produces good quality parameterization results for any open quadrilateral mesh that can be mapped to the plane; (2) The method minimizes the distortion of both angle and area caused by the parameterization; (3) The solution does not place any restrictions on the boundary shape; (4) Since the mapping computation is local, the method is faster and more efficient.

The rest of this paper is as follows. In section 2, we present our model and describe algorithm in detail. In Section 3,we give the results of our algorithm and compare them with other parameterization results. Finally, we draw conclusions in Section 4.

2 Parameterization

In this section, we outline the parameterization process of the quadrilateral mesh on the plane. Firstly, we reserve the boundary and the vertices with high Gaussian curvature and delete the interior vertices with low Gaussian curvature to simplify quadrilateral meshes. Secondly, the simplified mesh is mapped on the 2D domain through global parameterization. Thirdly, the weighted discrete mapping is used to embed the deleted vertices on the parameterization plane in angle-preserving and area-preserving manner, which will minimize the angle and area distortion. Finally, we optimize the parameterization mesh to eliminate the overlapping.

In the quadrilateral mesh, the one-ring neighbouring vertices of vertex v are the vertices sharing the same faces with the vertex v. The one-ring vertices of vertex v include two parts: immediate vertices and diagonal vertices.

2.1 Simplification Algorithm

The computation and the distortion may be large if the whole quadrilateral mesh is parameterized onto the plane. To speed up the parameterization and minimize the distortion, we reduce the number of vertices while trying to retain the good approximation of the original shape and appearance. The discrete curvature is one of the good criteria of simplification to preserve the shape of an original model.

In spite of the extensive use of quadrilateral mesh in geometric modeling and computer graphics, there is no agreement on the most appropriate way to estimate geometric attributes such as curvatures on discrete surfaces. Many surface-oriented applications need an approximation of the first and second order differential properties. Unfortunately, since meshes are piecewise linear surfaces, the concept of continuous curvature is not common. Thinking of a quadrilateral mesh as a piecewise linear approximation of an unknown smooth surface, we can try to estimate the curvature using only the information that is given by the quadrilateral mesh itself, such as the angles and faces that are connected to that vertex.

We estimate the discrete Gaussian curvature to reduce the number of the vertices and need not to compute the curvature accurately. The Gaussian curvature of vertex v is mainly determined by the immediate vertices, to speed up the computation we ignore the effect of the diagonal vertices and estimate the Gaussian curvature of quadrilateral meshes as shown in Fig.1-(a). We define integral Gaussian curvature $K = K_v$ with respect to the area $S = S_v$ attributed to v by

$$\overline{K} = \int_s K = 2\pi - \sum_{i=1}^{n} \theta_i \tag{1}$$

where θ_i is the angle between two successive edges. To derive the curvature at the vertex v from these integral values, we assume the curvature to be uniformly distributed around the vertex and simply normalized by the area.

$$K = \frac{\overline{K}}{S} \tag{2}$$

where S is the sum of the areas of adjacent faces around the vertex v. Different ways of defining the area S result in different curvature values. We use the Voronoi area, which sums up the areas of vertex v's local Voronoi cells restricted to the triangles adjacent to v, according to the Euclidean distance to the vertices of the mesh.

To determine the areas of the local Voronoi cells restricted to a triangle, we distinguish between obtuse and non-obtuse triangles in Fig.1. In the latter case they are given by

$$S_A = \frac{1}{8}(\left|v_i v_k\right|^2 \cot(\gamma_i) + \left|v_i v_j\right|^2 \cot(\delta_i)) \tag{3}$$

For obtuse triangles,

$$S_B = \frac{1}{8}\left|v_i v_k\right|^2 \tan(\gamma_i),$$

$$S_C = \frac{1}{8}\left|v_i v_j\right|^2 \tan(\delta_i),$$

$$S_A = S - S_B - S_{C.} \tag{4}$$

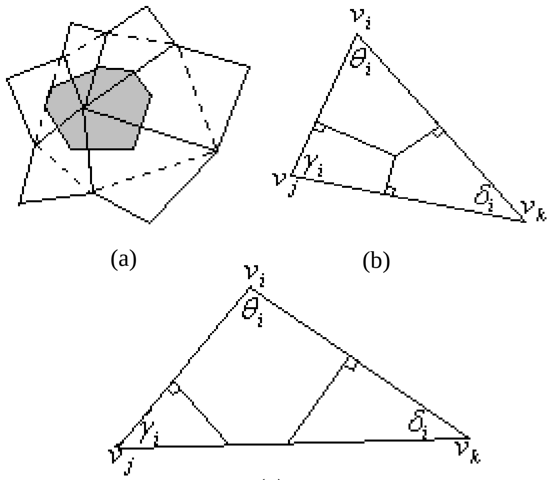


Figure 1. Voronoi area (a) Voronoi cells around a vertex (b) Non-obtus angle (c) Obtus angle

A vertex deletion means the deletion of the vertex with low Gaussian curvature and the edges that the vertex belongs to. To keep the geometry feature of the original mesh, the simplification algorithm is as follows:

1. Our simplification algorithm reads an input original quadrilateral mesh.
2. It calculates the discrete Gaussian curvature for all vertices using Eq.(1),(2),(3) or (4).
3. If the curvature is bigger than a user-given value, then our algorithm deletes the vertex and the correlative edges. Otherwise, it ends the simplification process.
4. Our algorithm outputs the simplified mesh, which is the polygonal mesh M .

During the simplification process, we can adjust the given value to change the simplified vertex number.

2.2 Global Parameterization

The vertices with low Gaussian curvature and the edges correlative with the vertices of the original mesh are deleted, and then a polygonal simplified mesh forms instead of a quadrilateral one. Parameterizing a 3D simplified mesh amounts to computing a correspondence between a discrete patch and an isomorphic planar mesh through a piecewise linear mapping. Given a piecewise simplified mesh M , construct a mapping between the mesh M and an isomorphic planar mesh $U \in R^2$ that best preserves the original, intrinsic characteristic. We denote by v_i the 3D position of i^{th} vertex in the mesh M , and by u_i the 2D position (parameterization value) of the corresponding vertex in the 2D mesh U .

The simplified polygonal mesh approximates the original quadrilateral mesh, but the angle and area are different from the original mesh. We take the edges of the mesh M as the spring and project the vertices of the mesh onto the parameterization domain by minimizing the edge-based energy function in the following

$$E_{global} = \frac{1}{2}\sum_{\{i,j\}\in Edge}\frac{1}{\left|v_i - v_j\right|^r}\left|u_i - u_j\right|^2, r \geq 0 \quad (5)$$

where $Edge$ is the edge set of the simplified mesh M ,the coefficients can be chosen in different ways according to r .

The parameterization approach reduces the number of vertices, so it is different from the previous global parameterization, which is more complicated and has more distortion. Besides, compared with fixed-boundary parameterization, the simplified mesh reserves the vertices with high Gaussian curvature, so parameterization results preserve the whole shape better.

2.3 Local Parameterization

After the boundary and the interior vertices with high Gaussian curvature are mapped on the 2D plane, the deleted vertices with low curvature should be embedded on the parameterization plane. The mapping of the interior vertices affects the parameterization results greatly, so we want to preserve as much of the intrinsic qualities of a mesh as we possibly can. This implies that we need to firstly define what these intrinsic qualities are for a discrete mesh: minimal distortion means best preservation of these qualities.

2.3 .1 Discrete Conformal Mapping

Conformal parameterization preserves angular structure, and is intrinsic to geometry and stable with respect to small deformations. To flatten a mesh to a two-dimensional plane so that it minimizes the relative distortion of the planar angles with respect to their counterparts in the three-dimensional space, we introduce an angle-based energy function as follows

$$E_A = \sum_{j\in N(i)}(\cot\frac{\alpha_{ij}}{4} + \cot\frac{\beta_{ij}}{4})\left|u_i - u_j\right|^2 \quad (6)$$

where $N(i)$ is the set of the immediate one-ring vertices of the vertex v_i , and α_{ij}, β_{ij} are the opposite left angles in the 3D mesh as shown in Fig.2-(a). The coefficients in the formula (6) are always positive, which reduces the overlapping of the parameterization mesh.

This is discrete quadratic energy in the parameterization and only depends on the angles of the original surface. To minimize the discrete conformal energy E_A, we get

$$\frac{\partial E_A}{\partial u_i} = 2\sum_{j\in N(i)} (\cot\frac{\alpha_{ij}}{4} + \cot\frac{\beta_{ij}}{4})(u_i - u_j) = 0 \quad (7)$$

From Equ. (7), we can get linear coefficients which are the functions of the angles of the original mesh.

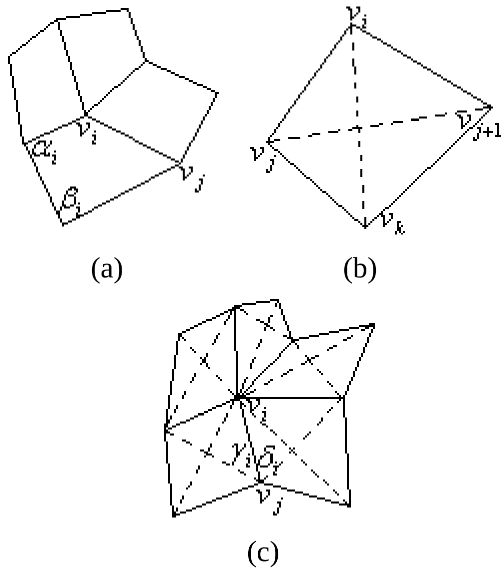


Figure 2. Edge and angles. (a) Edge and opposite left angles in the conformal mapping; (b) Quadrilateral mesh divided into four triangles (c) Edge and corresponding angles in the authalic mapping.

2.3.2 Discrete Authalic Mapping

Authalic mapping preserves the area as much as possible. The quadrilateral mesh in the 3D space is not flat, so we can not get the accurate area of each quadrilateral patch. To minimize the area distortion, we divide each quadrilateral patch into four triangular parts and then preserve the triangular areas differently. As shown in Fig.2-(b), the quadrilateral mesh $v_i v_j v_k v_{j+1}$ is divided into $\Delta v_i v_j v_{j+1}$, $\Delta v_i v_j v_k$, $\Delta v_i v_k v_{j+1}$, $\Delta v_j v_k v_{j+1}$. This will change the problem of quadrilateral area preserving into that of triangular area preserving.

The mapping resulted from the energy minimization has the property of preserving the area among each vertex's one-ring neighborhood of the mesh, and can be written as the form of

$$E_x = \sum_{j\in N(i)} \frac{(\cot\frac{\gamma_{ij}}{2} + \cot\frac{\delta_{ij}}{2})}{\left|v_i - v_j\right|^2} \left|u_i - u_j\right|^2 \qquad (8)$$

where γ_{ij} and δ_{ij} are corresponding angles of the edge (v_i, v_j) as shown in Fig.2-(c). The parameterization deriving from E_x is easily obtained when the parameterization value u_i satisfies

$$\frac{\partial E_x}{\partial u_i} = 2\sum_{j\in N(i)} \frac{(\cot\frac{\gamma_{ij}}{2} + \cot\frac{\delta_{ij}}{2})}{\left|v_i - v_j\right|^2}(u_i - u_j) = 0 \quad (9)$$

The way to solve this linear system is similar to that of discrete conformal mapping, but the linear coefficients are the functions of the local areas of the 3D mesh.

2.3.3 Weighted Discrete Parameterization

Discrete conformal mapping can be seen as an angle preserving mapping which minimizes the angle distortion for the interior vertices. The resulting mapping will preserve the shape but not the area of the original mesh. If we map a checkerboard image on the parameterization, the resulting texture mapped mesh will have the square of different sizes as shown in Fig3-(b),(f).

Discrete authlic mapping is area preserving which minimizes the area distortion. Although the area of the original mesh would locally be preserved, the shape tends to be distorted since the mapping from 3D to 2D will in general generate distortion. If we map a checkerboard image on the parameterization, the resulting texture mapping will have squares whose shape is distorted while local texture has the same size as shown in Fig3-(d),(h).

To minimize the distortion and get better parameterization results, we define linear combinations of the area and the angle distortion as the distortion measures. Therefore, it results that the family of admissible, simple distortion measures is reduced to linear combinations of the two discrete distortion measures defined above. A general distortion measure E as we define can thus always be written as

$$E = qE_A + (1-q)E_x \qquad (10)$$

where $q(0 \le q \le 1)$ is an real constant. By adjusting the scaling factor q, parameterizations appropriate for special applications can be got.

2.4 Mesh Optimization

The quadrilateral mesh is not restricted, such as convexity, so when parameterizing the mesh on the 2D plane, the parameterization mesh may produce overlapping. To eliminate the overlapping, we optimize the parameterization mesh by adjusting vertex location without changing the topology of the mesh. Mesh optimization is a local iterative process. Each vertex is optimized for the new location in a number of iterations.

Let u_i^q be q^{th} times iteration location of the parameterization value u_i, the optimization process to find the new location in iterations is described by the following formula

$$u_i^q = u_i^{q-1} + \lambda_1 \sum_{j=1}^{n} \left(\frac{u_j^{q-1} - u_i^{q-1}}{n}\right) + \lambda_2 \sum_{k=1}^{n} \left(\frac{u_k^{q-1} - u_i^{q-1}}{n}\right), 0 < \lambda_1 + \lambda_2 < 1 \quad (11)$$

where n is the degree of the vertex u_i, and u_j, u_k are the immediate and diagonal one-ring vertices respectively.

It is found that vertex optimization in order of "worst one first" is very helpful. We define the priority of the vertex u_i as the following

$$\sigma = \left| \lambda_1 \sum_{j=1}^{n} \left(\frac{u_j^{q-1} - u_i^{q-1}}{n}\right) + \lambda_2 \sum_{k=1}^{n} \left(\frac{u_k^{q-1} - u_i^{q-1}}{n}\right) \right| \quad (12)$$

The priority is simply computed based on the shape metrics of each parameterization vertex. For the vertex with the worst quality, the highest priority is assigned. Through experiments, we find that more times iterations are needed if vertices are not overlapped in an order of "first come first serve". Besides, we must point out that the optimization is local and we only optimize the overlapping vertices and its one-ring vertices, which will minimize the distortion and preserve the parameterization results better.

3.Examples

For a quantitative evaluation of various mesh parameterization methods we consider angle and area distortion error functions defined below. To measure the angle distortion error we use

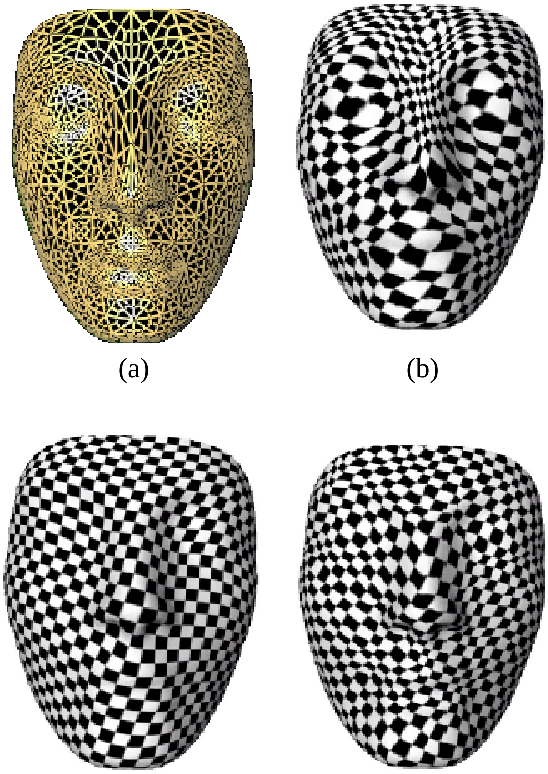
$$\sum_i \left(\frac{\theta_i}{\phi_i} - 1\right)^2 \quad (13)$$

where θ_i and ϕ_i are the corresponding angles of the original quadrilateral mesh and parameterization mesh respectively. The area distortion is measured by

$$\sum_j \sum_{i=1}^{4} \left(\frac{A(T_{j,i})}{A(U_{j,i})} - 1\right)^2 \quad (14)$$

where $A(T_{j,i}), A(U_{j,i})$ are the corresponding areas of the triangles that each quadrilateral patch is divided into.

To evaluate the visual quality of a parameterization we use the checkerboard texture shown in Fig.3, where the effect of the scaling factor q in Eq. (10) can be found. The parameterization approaches are discrete conformal mapping, weighted discrete mapping and discrete authalic mapping with $q = 0$, $q = 0.5$ and $q = 1$ respectively as shown in Fig.3. The angle and area distortions of the different mappings are shown in table 1.



(a) (b)

(c) (d)

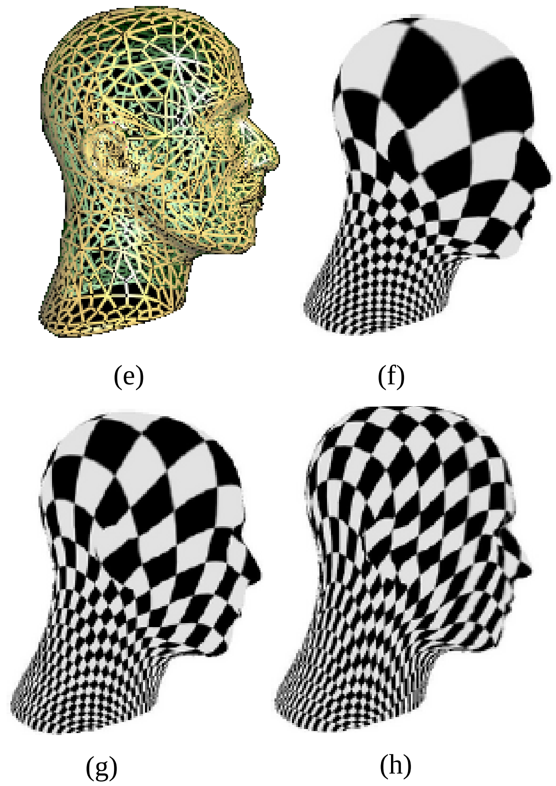


Figure 3. Texture mapping of the Face and Head Models with different mesh parameterizations. (a) and (e) Models; (b) and (f) Discrete conformal mapping , q=0; (c) and (g) Weighted discrete mapping , q=0.5; (d) and (h) Discrete Authalic mapping , q=1.

Table 1:Distortion energy of the parameterization

q	1	0.8	0.5	0.2	0
E_A -(a)	8.74	10.98	14.20	17.41	20.01
E_x -(a)	30.93	26.44	22.58	20.27	19.63
E_A -(e)	12.56	15.42	18.95	25.83	29.57
E_x -(e)	44.81	39.87	31.25	28.21	25.42

The results demonstrate that the medium q value (about 0.5) can get more smooth parameterization and minimal distortion energy of the parameterization. And the closer q to value 0 or 1, the larger the sum of the angle and area distortions is.

4.Conclusions

We present a parameterization approach for quadrilateral meshes based on mesh simplification and weighted discrete mapping. Mesh simplification reduces the computation, and the discrete mapping minimizes the angle and area distortion. The scaling factor q of the discrete mapping provides the flexibility for user to get appropriate parameterizations according to special applications, and establish different smoothness and distortion.

In future work, we will focus on using a better objective function to obtain better solutions and developing a good hierarchical solver that can speed up the mapping process, making parameterization of extremely large meshes tractable.

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