Matrix Based Subdivision Depth Computation for Extra-Ordinary Catmull-Clark Subdivision Surface Patches

Gang Chen and Fuhua (Frank) Cheng

Graphics & Geometric Modeling Lab, Department of Computer Science,
University of Kentucky, Lexington, Kentucky 40506-0046
{gchen5, cheng}@cs.uky.edu
www.cs.uky.edu/∼cheng

Abstract. A new subdivision depth computation technique for extra-ordinary Catmull-Clark subdivision surface (CCSS) patches is presented. The new technique improves a previous technique by using a matrix representation of the second order norm in the computation process. This enables us to get a more precise estimate of the rate of convergence of the second order norm of an extra-ordinary CCSS patch and, consequently, a more precise subdivision depth for a given error tolerance.

1 Introduction

Given a Catmull-Clark subdivision surface (CCSS) patch, *subdivision depth computation* is the process of determining how many times the control mesh of the CCSS patch should be subdivided so that the distance between the resulting control mesh and the surface patch is smaller than a given error tolerance.

A good subdivision depth computation technique requires precise estimate of the distance between the control mesh of a CCSS patch and its limit surface. Optimum distance evaluation techniques for regular CCSS patches are available [3,6]. Distance evaluation for an extra-ordinary CCSS patch is more complicated. A first attempt in that direction is done in [3]. The distance is evaluated by measuring norms of the *first order forward differences* of the control points. But the distance computed by this approach is usually bigger than what it really is for regions already flat enough and, consequently, leads to over-estimated subdivision depth.

An improved distance evaluation technique for extra-ordinary CCSS patches is presented in [4]. The distance is evaluated by measuring norms of the *second order forward differences* (called *second order norms*) of the control points of the given extra-ordinary CCSS patch. However, it has been observed recently that, for extra-ordinary CCSS patches, the convergence rate of second order norm changes with the subdivision process, especially between the first subdivision level and the second subdivision level. Therefore, using a fixed convergence rate in the distance evaluation process for all subdivision levels would over-estimate the distance and, consequently, over-estimate the subdivision depth as well.

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In this paper we present an improved subdivision depth computation method for extra-ordinary CCSS patches. The new technique uses a matrix representation of the maximum second order norm in the computation process to generate a recurrence formula. This recurrence formula allows the smaller convergence rate of the second subdivision level to be used as a bound in the evaluation of the maximum second order norm and, consequently, leads to a more precise subdivision depth for the given error tolerance.

2 Problem Formulation and Background

Given the control mesh of an extra-ordinary CCSS patch and an error tolerance ϵ, the goal here is to compute an integer d so that if the control mesh is iteratively refined (subdivided) d times, then the distance between the resulting mesh and the surface patch is smaller than ϵ. d is called the *subdivision depth* of the surface patch with respect to ϵ.

2.1 Distance Between Patch and Control Mesh

For a given interior mesh face $\mathbf{F}$, let $\mathbf{S}$ be the corresponding Catmull-Clark Subdivision Surface (CCSS) patch in the limit surface $\bar{\mathbf{S}}$. The *distance* between an interior mesh face $\mathbf{F}$ and the corresponding patch $\mathbf{S}$ is defined as the maximum of $\|\mathbf{F}(u,v) - \mathbf{S}(u,v)\|$:

$$D_{\mathbf{F}} = max_{(u,v)\in\Omega} \|\mathbf{F}(u,v) - \mathbf{S}(u,v)\| \tag{1}$$

where $\Omega \equiv [0,\ 1] \times [0,\ 1]$ is the parameter space of $\mathbf{F}$ and $\mathbf{S}$. $D_{\mathbf{F}}$ is also called the distance between $\mathbf{S}$ and its control mesh.

2.2 Depth Computation for Extra-Ordinary Patches

The distance evaluation mechanism of the previous subdivision depth computation technique for extra-ordinary CCSS patches utilizes second order norm as a measurement scheme as well [4], but the pattern of *second order forward differences* (SOFDs) used in the distance evaluation process is different from the one used for regular patches [4].

Second Order Norm and Recurrence Formula. Let $\mathbf{V}_i$, $i = 1, 2, ..., 2n+8$, be the control points of an extra-ordinary patch $\mathbf{S}(u,v) = \mathbf{S}_0^0(u,v)$, with $\mathbf{V}_1$ being an extra-ordinary vertex of valence n. The control points are ordered following J. Stam's fashion [7] (Figure 1(a)). The control mesh of $\mathbf{S}(u,v)$ is denoted $\Pi = \Pi_0^0$. The *second order norm* of $\mathbf{S}$, denoted $M = M_0$, is defined as the maximum norm of the following $2n + 10$ SOFDs:

$$\begin{aligned}
M = max\{ & \{\|2\mathbf{V}_1 - \mathbf{V}_{2i} - \mathbf{V}_{2((i+1)\%n+1)}\| \mid 1 \le i \le n\} \\
\cup\ & \{\|2\mathbf{V}_{2(i\%n+1)} - \mathbf{V}_{2i+1} - \mathbf{V}_{2(i\%n+1)+1}\| \mid 1 \le i \le n\} \\
\cup\ & \{\|2\mathbf{V}_3 - \mathbf{V}_2 - \mathbf{V}_{2n+8}\|,\ \|2\mathbf{V}_4 - \mathbf{V}_1 - \mathbf{V}_{2n+7}\|,\ \|2\mathbf{V}_5 - \mathbf{V}_6 - \mathbf{V}_{2n+6}\|, \\
& \ \|2\mathbf{V}_5 - \mathbf{V}_4 - \mathbf{V}_{2n+3}\|,\ \|2\mathbf{V}_6 - \mathbf{V}_1 - \mathbf{V}_{2n+4}\|,\ \|2\mathbf{V}_7 - \mathbf{V}_8 - \mathbf{V}_{2n+5}\|, \\
& \ \|2\mathbf{V}_{2n+7} - \mathbf{V}_{2n+6} - \mathbf{V}_{2n+8}\|,\ \|2\mathbf{V}_{2n+6} - \mathbf{V}_{2n+2} - \mathbf{V}_{2n+7}\|, \\
& \ \|2\mathbf{V}_{2n+3} - \mathbf{V}_{2n+2} - \mathbf{V}_{2n+4}\|,\ \|2\mathbf{V}_{2n+4} - \mathbf{V}_{2n+3} - \mathbf{V}_{2n+5}\|\ \}\ \}
\end{aligned} \tag{2}$$

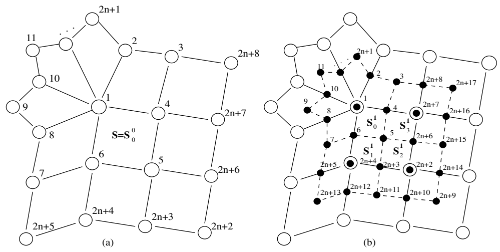

Fig. 1. (a) Ordering of control points of an extra-ordinary patch. (b) Ordering of new control points (solid dots) after a Catmull-Clark subdivision.

If we perform a Catmull-Clark subdivision step [1] on the control mesh of $\mathbf{S}$, we get four new subpatches: $\mathbf{S}_0^1$, $\mathbf{S}_1^1$, $\mathbf{S}_2^1$ and $\mathbf{S}_3^1$. $\mathbf{S}_0^1$ is an extra-ordinary patch but $\mathbf{S}_1^1$, $\mathbf{S}_2^1$ and $\mathbf{S}_3^1$ are regular patches (see Figure 1(b)). We use M_1 to denote the second order norm of $\mathbf{S}_0^1$. This process can be iteratively repeated on $\mathbf{S}_0^1$, $\mathbf{S}_0^2$, $\mathbf{S}_0^3$, ... etc. We have the following lemma for a general $\mathbf{S}_0^k$ and its second order norm M_k [4].

Lemma 1: For any $k \geq 0$, if M_k represents the second order norm of the extra-ordinary sub-patch $\mathbf{S}_0^k$ after k Catmull-Clark subdivision steps, then M_k satisfies the following inequality

$$M_{k+1} \leq \begin{cases} \frac{2}{3} M_k, & n = 3 \\ \frac{18}{25} M_k, & n = 5 \\ (\frac{3}{4} + \frac{8n-46}{4n^2}) M_k, & n > 5 \end{cases} .$$

Distance Evaluation. Let $\mathbf{L}(u,v)$ be the bilinear parametrization of the center face of $\mathbf{S}(u,v)$'s control mesh $\mathbf{F} = \{\mathbf{V}_1, \mathbf{V}_6, \mathbf{V}_5, \mathbf{V}_4\}$

$$\mathbf{L}(u,v) = (1-v)[(1-u)\mathbf{V}_1 + u\mathbf{V}_6] + v[(1-u)\mathbf{V}_4 + u\mathbf{V}_5], \quad 0 \leq u,v \leq 1$$

and let $\mathbf{S}(u,v)$ be parameterized following the Ω-partition based approach [7] then the maximum distance between $\mathbf{S}(u,v)$ and its control mesh satisfies the following lemma [4].

Lemma 2: The maximum of $\| \mathbf{L}(u,v) – \mathbf{S}(u,v) \|$ satisfies the following inequality

$$\| \mathbf{L}(u,v) - \mathbf{S}(u,v) \| \leq \begin{cases} M_0, & n = 3 \\ \frac{5}{7} M_0, & n = 5 \\ \frac{4n}{n^2 - 8n + 46} M_0, & 5 < n \leq 8 \\ \frac{n^2}{4(n^2 - 8n + 46)} M_0, & n > 8 \end{cases} \tag{3}$$

where $M = M_0$ is the second order norm of the extra-ordinary patch $\mathbf{S}(u,v)$.

Subdivision Depth Computation. Lemma 2 can be used to estimate the distance between a level-k control mesh and the surface patch for any $k > 0$.

Theorem 3: Given an extra-ordinary surface patch $\mathbf{S}(u,v)$ and an error tolerance ϵ, if k levels of subdivisions are iteratively performed on the control mesh of $\mathbf{S}(u,v)$, where $k = \lceil log_w \frac{M}{z\epsilon} \rceil$ with M being the second order norm of $\mathbf{S}(u,v)$ defined in (2),

$$w = \begin{cases} \frac{3}{2}, & n = 3 \\ \frac{25}{18}, & n = 5 \\ \frac{4n^2}{3n^2+8n-46}, & n > 5 \end{cases} \quad \text{and} \quad z = \begin{cases} 1, & n = 3 \\ \frac{25}{18}, & 5 \le n \le 8 \\ \frac{2(n^2-8n+46)}{n^2}, & n > 8 \end{cases}$$

then the distance between $\mathbf{S}$ and the level-k control mesh is smaller than ϵ.

3 New Subdivision Depth Computation Technique for Extra-Ordinary Patches

The SOFDs involved in the second order norm of an extra-ordinary CCSS patch (see eq. (2)) can be classified into two groups: *group I* and *group II*. Group I contains those SOFDs that involve vertices in the vicinity of the extra-ordinary vertex (see Figure 2(a)). These are the first $2n$ SOFDs in (2). Group II contains the remaining SOFDs, i.e., SOFDs that involve vertices in the vicinity of the other three vertices of $\mathbf{S}$ (see Figure 2(b)). These are the last 10 SOFDs in (2). It is easy to see that the convergence rate of the SOFDs in group II is the same as the regular case, i.e., 1/4 [3]. Therefore, to study properties of the second order norm M, it is sufficient to study norms of the SOFDs in group I.

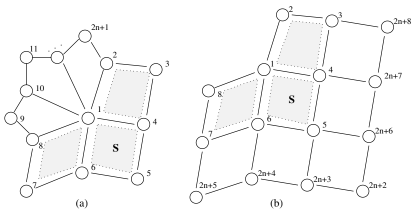

Fig. 2. (a) Vicinity of the extra-ordinary point. (b) Vicinity of the other three vertices of $\mathbf{S}$.

3.1 Matrix Based Rate of Convergence

The second order norm of $\mathbf{S} = \mathbf{S}_0^0$ can be put in matrix form as follows:

$$M = \|\mathrm{A}\mathbf{P}\|_\infty$$

where A is a $2n * (2n+1)$ matrix

$$\mathrm{A} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 2 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & \cdots & 0 & 0 \\ & & & & & \vdots & & & & & \\ 2 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & \cdots & -1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ & & & & & \vdots & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 2 & -1 \end{bmatrix}$$

and $\mathbf{P}$ is a control point vector

$$\mathbf{P} = [\mathbf{V}_1,\ \mathbf{V}_2,\ \mathbf{V}_3, \ldots,\ \mathbf{V}_{2n+1}]^T.$$

A is called the *second order norm matrix* for extra-ordinary CCSS patches. If i levels of Catmull-Clark subdivision are performed on the control mesh of $\mathbf{S} = \mathbf{S}_0^0$ then, following the notation of Section 2, we have an extra-ordinary subpatch $\mathbf{S}_0^i$ whose second order norm can be expressed as:

$$M_i = \left\| \mathrm{A}\Lambda^i \mathbf{P} \right\|_\infty$$

where Λ is a subdivision matrix of dimension $(2n+1)*(2n+1)$. The function of Λ is to perform a subdivision step on the $2n+1$ control vertices around (and including) the extra-ordinary point (see Figure 2(a)). We are interested in knowing the relationship between $\|\mathrm{A}\mathbf{P}\|_\infty$ and $\left\|\mathrm{A}\Lambda^i\mathbf{P}\right\|_\infty$. We need the following important result for $\mathrm{A}\Lambda^i$. The proof of this result is shown in [2].

Lemma 4: $\mathrm{A}\Lambda^i = \mathrm{A}\Lambda^i\mathrm{A}^+\mathrm{A}$, where A^+ is the *pseudo-inverse matrix* of A.

With this lemma, we have

$$\frac{\left\|\mathrm{A}\Lambda^i\mathbf{P}\right\|_\infty}{\|\mathrm{A}\mathbf{P}\|_\infty} = \frac{\left\|\mathrm{A}\Lambda^i\mathrm{A}^+\mathrm{A}\mathbf{P}\right\|_\infty}{\|\mathrm{A}\mathbf{P}\|_\infty} \le \frac{\left\|\mathrm{A}\Lambda^i\mathrm{A}^+\right\|_\infty \|\mathrm{A}\mathbf{P}\|_\infty}{\|\mathrm{A}\mathbf{P}\|_\infty} = \left\|\mathrm{A}\Lambda^i\mathrm{A}^+\right\|_\infty$$

Use r_i to represent $\left\|\mathrm{A}\Lambda^i\mathrm{A}^+\right\|_\infty$. Then we have the following recurrence formula for r_i

$$\begin{aligned} r_i \equiv \left\|\mathrm{A}\Lambda^i\mathrm{A}^+\right\|_\infty &= \left\|\mathrm{A}\Lambda^{i-1}\mathrm{A}^+\mathrm{A}\Lambda\mathrm{A}^+\right\|_\infty \\ &\le \left\|\mathrm{A}\Lambda^{i-1}\mathrm{A}^+\right\|_\infty \ \left\|\mathrm{A}\Lambda\mathrm{A}^+\right\|_\infty = r_{i-1}\ r_1 \end{aligned} \tag{4}$$

where $r_0 = 1$. Hence, we have the following lemma on the convergence rate of second order norm of an extra-ordinary CCSS patch.

Lemma 5: The second order norm of an extra-ordinary CCSS patch satisfies the following inquality:

$$M_i \le r_i\ M_0 \tag{5}$$

where $r_i = \left\|\mathrm{A}\Lambda^i\mathrm{A}^+\right\|_\infty$ and r_i satisfies the recurrence formula (4).

The recurrence formula (4) shows that r_i in (5) can be replaced with r_1^i. However, experiment data show that, while the convergence rate changes by a constant ratio in most of the cases, there is a significant difference between r_2 and r_1. The value of r_2 is smaller than r_1^2 by a significant gap. Hence, if we use r_1^i for r_i in (5), we would end up with a bigger subdivision depth for a given error tolerance. A better choice is to use r_2 to bound r_i, as follows.

$$r_i \le \begin{cases} r_2^j, & \mathrm{i} = 2j \\ r_1 r_2^j, & \mathrm{i} = 2j+1 \end{cases} \tag{6}$$

3.2 Distance Evaluation

Following (12) and (13) of [4], the distance between the extra-ordinary CCSS patch $\mathbf{S}(u,v)$ and its control mesh $\mathbf{L}(u,v)$ can be expressed as

$$\begin{aligned}&\|\mathbf{L}(u,v)-\mathbf{S}(u,v)\| \le \textstyle\sum_{k=0}^{m-2}\|\mathbf{L}_0^k(u_k,v_k)-\mathbf{L}_0^{k+1}(u_{k+1},v_{k+1})\| \\ &+\|\mathbf{L}_0^{m-1}(u_{m-1},v_{m-1})-\mathbf{L}_b^m(u_m,v_m)\|+\|\mathbf{L}_b^m(u_m,v_m)-\mathbf{S}_b^m(u_m,v_m)\|\end{aligned} \tag{7}$$

where u_m,v_m and b are defined in [4].

By applying Lemma 5, Lemma 6 and Lemma 1 of [4] on the first, second and third terms of the right hand side of the above inequality, respectively, we get

$$\begin{aligned}\|\mathbf{L}(u,v)-\mathbf{S}(u,v)\| &\le c\textstyle\sum_{k=0}^{m-2}M_k+\frac{1}{4}M_{m-1}+\frac{1}{3}M_m \\ &\le M_0(c\textstyle\sum_{k=0}^{m-2}r_k+\frac{1}{4}r_{m-1}+\frac{1}{3}r_m)\end{aligned}$$

where $c = 1/\min\{n,8\}$. The last part of the above inequality follows from Lemma 2. Consequently, through a simple algebra, we have

$$\|\mathbf{L}(u,v)-\mathbf{S}(u,v)\| \le \begin{cases} M_0[c(\frac{1-r_2^j}{1-r_2}+\frac{1-r_2^{j-1}}{1-r_2}r_1)+\frac{r_1r_2^{j-1}}{4}+\frac{r_2^j}{3}], & \text{if} \quad m=2j \\ M_0[c(\frac{1-r_2^j}{1-r_2}+\frac{1-r_2^j}{1-r_2}r_1)+\frac{r_2^j}{4}+\frac{r_1r_2^j}{3}], & \text{if} \quad m=2j+1 \end{cases}$$

It can be easily proved that the maximum occurs at $m=\infty$. Hence, we have the following lemma.

Lemma 6: The maximum of $\|\mathbf{L}(u,v)-\mathbf{S}(u,v)\|$ satisfies the following inequality

$$\|\mathbf{L}(u,v)-\mathbf{S}(u,v)\| \le \frac{M_0}{\min\{n,8\}}\frac{1+r_1}{1-r_2}$$

where $r_i = \|\mathrm{A}\Lambda^i\mathrm{A}^+\|_\infty$ and $M = M_0$ is the second order norm of the extra-ordinary patch $\mathbf{S}(u,v)$.

3.3 Subdivision Depth Computation

Lemma 6 can also be used to evaluate the distance between a level-i control mesh and the extra-ordinary patch $\mathbf{S}(u,v)$ for any $i>0$. This is because the distance between a level-i control mesh and the surface patch $\mathbf{S}(u,v)$ is dominated by the distance between the level-i extra-ordinary subpatch and the corresponding control mesh which, accoriding to Lemma 6, is

$$\|\mathbf{L}_i(u,v)-\mathbf{S}(u,v)\| \le \frac{M_i}{\min\{n,8\}}\frac{1+r_1}{1-r_2}$$

where M_i is the second order norm of $\mathbf{S}(u,v)$'s level-i control mesh, $\mathbf{M}_i$. Hence, if the right side of the above inequality is smaller than a given error tolerance ϵ, then the distance between $\mathbf{S}(u,v)$ and the level-i control mesh is smaller than ϵ. Consequently, we have the following subdivision depth computation theorem for extra-ordinary CCSS patches.

Theorem 7: Given an extra-ordinary surface patch $\mathbf{S}(u,v)$ and an error tolerance ϵ, if

$$i \equiv \min\{2l,\ 2k+1\}$$

levels of subdivision are iteratively performed on the control mesh of $\mathbf{S}(u,v)$, where

$$l = \lceil \log_{\frac{1}{r_2}} (\frac{1}{\min\{n,8\}} \frac{1+r_1}{1-r_2} \frac{M_0}{\epsilon}) \rceil \ ,$$
$$k = \lceil \log_{\frac{1}{r_2}} (\frac{r_1}{\min\{n,8\}} \frac{1+r_1}{1-r_2} \frac{M_0}{\epsilon}) \rceil$$

with $r_i = \|\mathrm{A}\Lambda^i \mathrm{A}^+\|_\infty$ and M_0 being the second order norm of $\mathbf{S}(u, v)$, then the distance between $\mathbf{S}(u, v)$ and the level-i control mesh is smaller than ϵ.

4 Examples

The new subdivision depth technique has been inplemented in *C++* on the Windows platform to compare its performance with the previous approach. *MatLab* is used for both numerical and symbolic computation of r_i in the implementation. Table 1 shows the comparison results of the previous technique, Theorem 3, with the new technique, Theorem 7. Two error tolerances 0.01 and 0.001 are considered and the second order norm M_0 is assumed to be 2. For each error tolerance, we consider five different valences: 3, 5, 6, 7 and 8 for the extra-ordinary vertex. As can be seen from the table, the new technique has a 30% improvement over the previous technique in most of the cases. Hence, the new technique indeed improves the previous technique significantly.

To show that the rates of convergence are indeed difference between r_1 and r_2, their values from several typical extra-ordinary CCSS patches are also included in Table 1. Note that when we compare r_1 and r_2, the value of r_1 should be squared first.

Table 1. Comparison between the old and the new technique

	$\epsilon = 0.01$		$\epsilon = 0.001$		convergence rate	
N	Old	New	Old	New	r_1	r_2
3	14	9	19	12	0.6667	0.2917
5	16	11	23	16	0.7200	0.4016
6	19	16	27	22	0.8889	0.5098
7	23	14	33	22	0.8010	0.5121
8	37	27	49	33	1.0078	0.5691

5 Conclusions

A new subdivision depth computation technique for extra-ordinary CCSS patches is presented. The computation process is performed on matrix representation of the second order norm, which gives us a better bound of the convergence rate and, consequently, a tighter subdivision depth for a given error tolerance. Test results show that the new technique improves the previous technique by about 30% in most of the cases. This is a significant result because of the exponential nature of the subdivision process.

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References

1. Catmull E, Clark J, Recursively Generated B-spline Surfaces on Arbitrary Topological Meshes, *Computer-Aided Design 10*, 6, 350-355, 1978.
2. Chen G, Cheng F, Matrix based Subdivision Depth Computation for Extra-Ordinary Catmull-Clark Subdivision Surface Patches (complete version), http://www.cs.uky.edu/∼cheng/PUBL/ sub_depth_3.pdf
3. Cheng F, Yong J, Subdivision Depth Computation for Catmull-Clark Subdivision Surfaces, *Computer Aided Design & Applications 3*, 1-4, 2006.
4. Cheng F, Chen G, Yong J, Subdivision Depth Computation for Extra-Ordinary Catmull-Clark Subdivision Surface Patches, to appear in *Lecture Notes in Computer Science*, Springer, 2006.
5. Halstead M, Kass M, DeRose T, Efficient, Fair Interpolation Using Catmull-Clark Surfaces, *Proceedings of SIGGRAPH 1993*, 35-44.
6. Lutterkort D, Peters J, Tight linear envelopes for splines, *Numerische Mathematik 89*, 4, 735-748, 2001.
7. Stam J, Exact Evaluation of Catmull-Clark Subdivision Surfaces at Arbitrary Parameter Values, *Proceedings of SIGGRAPH 1998*, 395-404.