Matrix based Subdivision Depth Computation for Extra-Ordinary Catmull-Clark Subdivision Surface Patches

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Abstract

A new subdivision depth computation technique for extra-ordinary Catmull-Clark subdivision surface (CCSS) patches is presented. The new technique improves a previous technique by using a matrix representation of the second order norm in the computation process. This enables us to get a more precise estimate of the rate of convergence of the second order norm of an extra-ordinary CCSS patch and, consequently, a more precise subdivision depth for a given error tolerance.

Keywords: subdivision surfaces, subdivision depth computation

1 Introduction

Given a Catmull-Clark subdivision surface (CCSS) patch, *subdivision depth computation* is the process of determining how many times the control mesh of the CCSS patch should be subdivided so that the distance between the resulting control mesh and the surface patch is smaller than a given error tolerance. Good subdivision depth computation techniques are important because they allows us to meet precision requirement in applications such as *trimming*, *finite element mesh generation*, *boolean operations*, and *tessellation* of a CCSS without excessively subdividing its control mesh.

A good subdivision depth computation technique requires precise estimate of the distance between the control mesh and the limit surface. Optimum distance evaluation techniques for regular CCSS patches are available [4, 11]. Distance evaluation for an extra-ordinary CCSS patch is more complicated. A first attempt in that direction is done in [4]. The distance is evaluated by measuring norms of the *first order forward differences* of the control points. Since first order forward differences can not measure the curvature of a surface but its dimension, the distance computed by this approach is usually bigger than what it really is for regions already flat enough and, consequently, leads to over-estimated subdivision depth.

An improved distance evaluation technique for extra-ordinary CCSS patches is presented in [5]. The distance is evaluated by measuring norms of the *second order forward differences* (called *second order norms*) of the control points of the given extra-ordinary CCSS patch. Since second order forward differences can measure both height and width of a region, the distance computed by this approach reflects curvature of the patch and, hence, leads to reasonable subdivision depths for regions already flat enough. However, it has been observed recently that, for extra-ordinary CCSS patches, the convergence rate of second order norm changes with the subdivision process, especially between the first subdivision level and the second subdivision level. Therefore, using a fixed convergence rate in the distance evaluation process for all subdivision levels would over-estimate the distance and, consequently, over-estimate the subdivision depth as well.

In this paper we present an improved subdivision depth computation method for extra-ordinary CCSS patches. The new technique uses a matrix representation of the maximum second order norm in the computation process to generate a recurrence formula. This recurrence formula allows the smaller convergence rate of the second subdivision level to be used as a bound in the evaluation of the maximum second order norm and, consequently, leads to a more precise subdivision depth for the given error tolerance.

The remaining part of the paper is arranged as follows. A brief review of the background is given in Section 2. A matrix based subdivision depth computation technique for extra-ordinary CCSS patches is presented in section 3. Examples showing the new technique improves the old one are presented in Section 4. Concluding remarks are given in Section 5.

2 Problem Formulation and Background

Given a control mesh $\mathbf{M} = \mathbf{M}_0$, let $\bar{\mathbf{S}}$ be its Catmull-Clark subdivision surface (CCSS). For each interior face $\mathbf{F}$ of $\mathbf{M}$, there is a corresponding patch $\mathbf{S}$ in the limit surface $\bar{\mathbf{S}}$. The control mesh of $\mathbf{S}$ contains $\mathbf{F}$ as the center face. If we perform a Catmull-Clark subdivision step on the control mesh, we get four new mesh faces in the place of $\mathbf{F}$. This is the case no matter $\mathbf{F}$ is a regular face or an extra-ordinary face. See Figure 1(b) for the four new faces $\mathbf{F}_{00}$, $\mathbf{F}_{10}$, $\mathbf{F}_{01}$ and $\mathbf{F}_{11}$ in the place of the extra-ordinary face $\mathbf{F}$ shown in Figure 1(a). Since each of these new faces corresponds to a quarter subpatch of $\mathbf{S}$, we shall call these new faces *subfaces* of $\mathbf{F}$ even though they are not pyhsically subsets of $\mathbf{F}$. Therefore, each subdivision step generates four new subfaces for the center face $\mathbf{F}$ of the control mesh. Because the correspondence between $\mathbf{F}$ and $\mathbf{S}$ is one-to-one, sometime, instead of saying performing a subdivision step on $\mathbf{S}$, we simply say performing a subdivision step on $\mathbf{F}$.

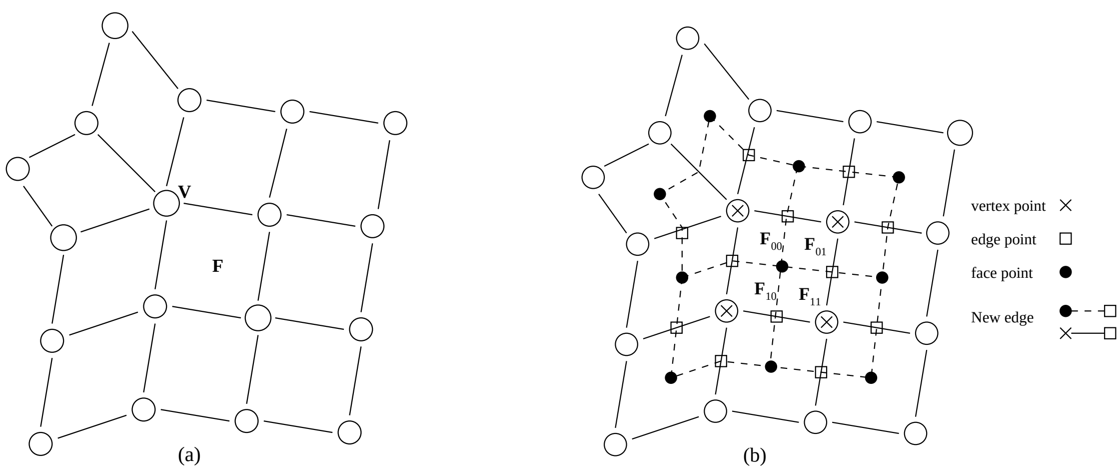

Figure 1: (a) Control mesh of an extra-ordinary patch; (b) new vertices and edges generated after a Catmull-Clark subdivision.

The *distance* between an interior mesh face $\mathbf{F}$ and the corresponding patch $\mathbf{S}$ is defined as the maximum of $\|\mathbf{L}(u, v) - \mathbf{S}(u, v)\|$:

$$D_{\mathbf{F}} = max_{(u,v)\in\Omega} \|\mathbf{L}(u, v) - \mathbf{S}(u, v)\| \tag{1}$$

where Ω is the unit square parameter space of $\mathbf{S}$ and $\mathbf{L}(u, v)$ is the bilinear parametrization of $\mathbf{F}$ on Ω. $D_{\mathbf{F}}$ is also called the distance between $\mathbf{S}$ and its control mesh. For a given $\epsilon > 0$, the *subdivision depth* of $\mathbf{F}$ with respect to ϵ is a positive integer d such that if $\mathbf{F}$ is recursively subdivided d times, the distance between each of the resulting subfaces and the corresponding subpatch is smaller than ϵ. In the following, we review some of the previous results needed in the new work.

2.1 Distance Evaluation for a Regular Patch

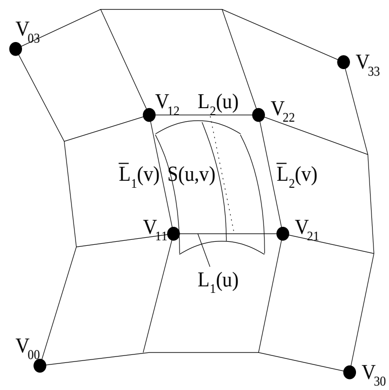

Figure 2: Definition of $\mathbf{L}(u,v) = (1-v)\mathbf{L}_1(u) + v\mathbf{L}_2(u) = (1-u)\bar{\mathbf{L}}_1(v) + u\bar{\mathbf{L}}_2(v)$.

Let $\mathbf{S}(u,v)$ be a uniform bicubic B-spline surface patch defined on the unit square $\Omega = [0,1] \times [0,1]$ with control points $\mathbf{V}_{i,j}$, $0 \le i,j \le 3$, and let $\mathbf{L}(u,v)$ be the bilinear parametrization of the center mesh face $\{\mathbf{V}_{1,1}, \mathbf{V}_{2,1}, \mathbf{V}_{2,2}, \mathbf{V}_{1,2}\}$ (see Figure 2):

$$\mathbf{L}(u,v) = (1-v)[(1-u)\mathbf{V}_{1,1} + u\mathbf{V}_{2,1}] + v[(1-u)\mathbf{V}_{1,2} + u\mathbf{V}_{2,2}], \quad 0 \le u,v \le 1.$$

Then the distance between $\mathbf{S}(u,v)$ and $\mathbf{L}(u,v)$ satisfies the following lemma [4].

Lemma 1: The distance between $\mathbf{L}(u,v)$ and $\mathbf{S}(u,v)$ satisfies the following inequality

$$\max_{0 \le u,v \le 1} \|\mathbf{L}(u,v) - \mathbf{S}(u,v)\| \le \frac{1}{3} M$$

where M is the *second order norm* of $\mathbf{S}(u,v)$ defined as follows

$$M = \max_{i,j}\{ \ \|2\mathbf{V}_{i,j} - \mathbf{V}_{i-1,j} - \mathbf{V}_{i+1,j}\| \ , \ \ \|2\mathbf{V}_{i,j} - \mathbf{V}_{i,j-1} - \mathbf{V}_{i,j+1}\| \ \} \tag{2}$$

2.2 Subdivision Depth Computation for Extra-Ordinary Patches

The distance evaluation mechanism of the previous subdivision depth computation technique for extra-ordinary CCSS patches utilizes second order norm as a measurement scheme as well [5], but the pattern of *second order forward differences* (SOFDs) used in the distance evaluation process is different from (2).

Let $\mathbf{V}_i$, $i = 1, 2, ..., 2n+8$, be the control points of an extra-ordinary patch $\mathbf{S}(u,v) = \mathbf{S}_0^0(u,v)$, with $\mathbf{V}_1$ being an extra-ordinary vertex of valence n. The control points are ordered following J. Stam's fashion [14] (Figure 3(a)). The control mesh of $\mathbf{S}(u,v)$ is denoted $\Pi = \Pi_0^0$. The *second order norm* of $\mathbf{S}$, denoted $M = M_0$, is defined as the maximum norm of the following SOFDs. There are $2n+10$ of them.

$$\begin{aligned} M = max\{ \ &\{ \ \|2\mathbf{V}_1 - \mathbf{V}_{2i} - \mathbf{V}_{2((i+1)\%n+1)}\| \mid 1 \le i \le n\} \ \cup \ \{ \ \|2\mathbf{V}_{2(i\%n+1)} - \mathbf{V}_{2i+1} - \mathbf{V}_{2(i\%n+1)+1}\| \mid 1 \le i \le n\} \\ &\cup \ \{ \ \| \ 2\mathbf{V}_3 - \mathbf{V}_2 - \mathbf{V}_{2n+8} \ \|, \ \| \ 2\mathbf{V}_4 - \mathbf{V}_1 - \mathbf{V}_{2n+7} \ \|, \ \| \ 2\mathbf{V}_5 - \mathbf{V}_6 - \mathbf{V}_{2n+6} \ \|, \ \| \ 2\mathbf{V}_{2n+3} - \mathbf{V}_{2n+2} - \mathbf{V}_{2n+4} \ \|, \\ &\quad \| \ 2\mathbf{V}_7 - \mathbf{V}_8 - \mathbf{V}_{2n+5} \ \|, \ \| \ 2\mathbf{V}_6 - \mathbf{V}_1 - \mathbf{V}_{2n+4} \ \|, \ \| \ 2\mathbf{V}_5 - \mathbf{V}_4 - \mathbf{V}_{2n+3} \ \|, \ \| \ 2\mathbf{V}_{2n+6} - \mathbf{V}_{2n+2} - \mathbf{V}_{2n+7} \ \|, \\ &\quad \| \ 2\mathbf{V}_{2n+7} - \mathbf{V}_{2n+6} - \mathbf{V}_{2n+8} \ \|, \ \| \ 2\mathbf{V}_{2n+4} - \mathbf{V}_{2n+3} - \mathbf{V}_{2n+5} \ \| \ \} \ \} \end{aligned} \tag{3}$$

By performing a subdividion step on Π, one gets $2n+17$ new vertices $\mathbf{V}_i^1$, $i = 1, ..., 2n+17$ (see Figure 3(b)). These control points form four control point sets Π_0^1, Π_1^1, Π_2^1 and Π_3^1, representing control meshes

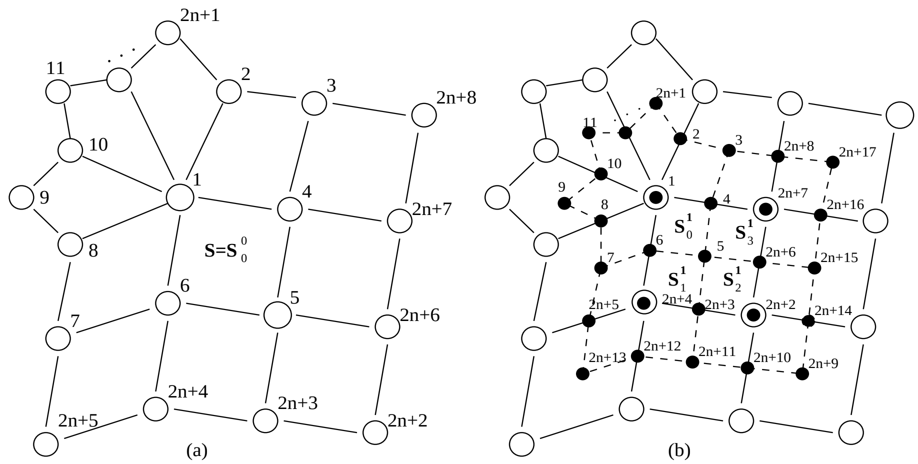

Figure 3: (a) Ordering of control points of an extra-ordinary patch. (b) Ordering of new control points (solid dots) after a Catmull-Clark subdivision.

of the subpatches $\mathbf{S}_0^1$, $\mathbf{S}_1^1$, $\mathbf{S}_2^1$ and $\mathbf{S}_3^1$, respectively (see Figure 3(b)) where $\Pi_0^1 = \{\mathbf{V}_i^1 \mid 1 \le i \le 2n+8\}$, and the other three control point sets Π_1^1, Π_2^1 and Π_3^1 are shown in Figure 4. $\mathbf{S}_0^1$ is an extra-ordinary patch but $\mathbf{S}_1^1$, $\mathbf{S}_2^1$ and $\mathbf{S}_3^1$ are regular patches. Therefore, second order norm similar to the one defined in (2) can be defined for $\mathbf{S}_1^1$, $\mathbf{S}_2^1$ and $\mathbf{S}_3^1$, while a second order norm similar to (3) can be defined for the control mesh

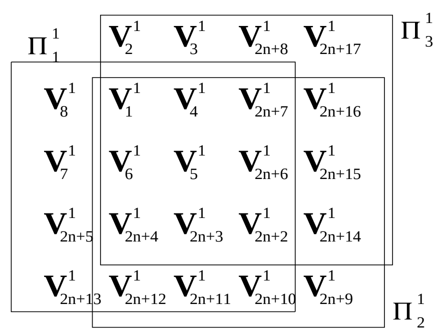

Figure 4: Control vertices of subpatches $\mathbf{S}_1^1$, $\mathbf{S}_2^1$ and $\mathbf{S}_3^1$.

of $\mathbf{S}_0^1$. We use M_1 to denote the second order norm of $\mathbf{S}_0^1$. This process can be iteratively repeated on $\mathbf{S}_0^1$, $\mathbf{S}_0^2$, $\mathbf{S}_0^3$, ... etc. We have the following lemma for a general $\mathbf{S}_0^k$ and its second order norm M_k [5].

Lemma 2: For any $k \ge 0$, if M_k represents the second order norm of the extra-ordinary sub-patch $\mathbf{S}_0^k$ after k Catmull-Clark subdivision steps, then M_k satisfies the following inequality

$$M_{k+1} \le \begin{cases} \frac{2}{3} M_k, & n = 3 \\ \frac{18}{25} M_k, & n = 5 \\ (\frac{3}{4} + \frac{8n-46}{4n^2}) M_k, & n > 5 \end{cases} .$$

Actually, the lemma works in a more general sense, i.e., if M_k stands for the second order norm of the control mesh $\mathbf{M}_k$, instead of Π_0^k, the lemma still works. The second order norm of $\mathbf{M}_k$ is defined as follows: for regions not involving the extra-ordinary point, use standard SOFDs; for the vicinity of the extra-ordinary point, use SOFDs defined in (3). The proof is essentially the same.

2.2.1 Distance Evaluation

To compute the distance between the extra-ordinary patch $\mathbf{S}(u,v)$ and the center face of its control mesh, $\mathbf{F} = \{\mathbf{V}_1, \mathbf{V}_6, \mathbf{V}_5, \mathbf{V}_4\}$, we need to parameterize the patch $\mathbf{S}(u,v)$ first.

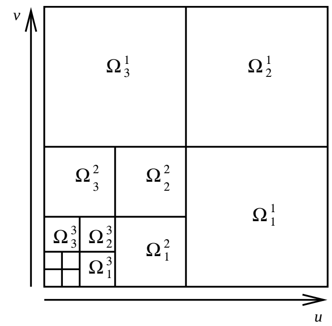

Figure 5: Ω-partition of the unit square.

By iteratively performing Catmull-Clark subdivision on $\mathbf{S}(u,v) = \mathbf{S}_0^0$, $\mathbf{S}_0^1$, $\mathbf{S}_0^2$, ... etc, we get a sequence of regular patches $\{ \mathbf{S}_b^m \}$, $m \geq 1$, $b = 1, 2, 3$, and a sequence of extra-ordinary patches $\{ \mathbf{S}_0^m \}$, $m \geq 1$. The extra-ordinary patches converge to a limit point which is the value of $\mathbf{S}$ at $(0,0)$ [8]. This limit point and the regular patches $\{ \mathbf{S}_b^m \}$, $m \geq 1$, $b = 1, 2, 3$, form a partition of $\mathbf{S}$. If we use Ω_b^m to represent the region of the parameter space that corresponds to $\mathbf{S}_b^m$ then $\{ \Omega_b^m \}$, $m \geq 1$, $b = 1, 2, 3$, form a partition of the unit square $\Omega = [0,1] \times [0,1]$ (see Figure 5) with

$$\Omega_1^m = [\frac{1}{2^m}, \frac{1}{2^{m-1}}] \times [0, \frac{1}{2^m}], \qquad \Omega_2^m = [\frac{1}{2^m}, \frac{1}{2^{m-1}}] \times [\frac{1}{2^m}, \frac{1}{2^{m-1}}], \qquad \Omega_3^m = [0, \frac{1}{2^m}] \times [\frac{1}{2^m}, \frac{1}{2^{m-1}}]. \tag{4}$$

The parametrization of $\mathbf{S}(u,v)$ is done as follows. For any $(u,v) \in \Omega$ but $(u,v) \neq (0,0)$, first find the Ω_b^m that contains (u,v). m and b can be computed as follows.

$$m(u,v) = \min\{\lceil log_{\frac{1}{2}} u \rceil,\ \lceil log_{\frac{1}{2}} v \rceil\}, \qquad b(u,v) = \begin{cases} 1, & if\ 2^m u \geq 1\ and\ 2^m v \leq 1 \\ 2, & if\ 2^m u \geq 1\ and\ 2^m v \geq 1 \\ 3, & if\ 2^m u \leq 1\ and\ 2^m v \geq 1 \end{cases} \tag{5}$$

Then map this Ω_b^m to the unit square with the following mapping

$$(u,v) \rightarrow\ (u_m, v_m)$$

where

$$t_m = (2^m t)\%1 = \begin{cases} 2^m t, & if\ \ 2^m t \leq 1 \\ 2^m t - 1, & if\ \ 2^m t > 1 \end{cases} \tag{6}$$

The value of $\mathbf{S}(u,v)$ is equal to the value of $\mathbf{S}_b^m$ at (u_m, v_m), i.e.,

$$\mathbf{S}(u,v) = \mathbf{S}_b^m(u_m, v_m).$$

Let $\mathbf{L}_b^m(u,v)$ be the bilinear parametrization of the center face of $\mathbf{S}_b^m$'s control mesh. Since $\mathbf{S}_b^m$ is a regular patch, following Lemma 1, we have

$$\|\mathbf{L}_b^m(u,v) - \mathbf{S}_b^m(u,v)\| \leq \frac{1}{3} M_b^m$$

where M_b^m is the second order norm of the contol mesh of $\mathbf{S}_b^m$. The second order norm of $\mathbf{S}_b^m$ is smaller than the second order norm of $\mathbf{M}_m$, M_m. Hence, the above inequality can be written as

$$\|\mathbf{L}_b^m(u,v) - \mathbf{S}_b^m(u,v)\| \leq \frac{1}{3} M_m. \tag{7}$$

If we use $\mathbf{L}(u,v)$ to represent the bilinear parametrization of the center face of $\mathbf{S}(u,v)$'s control mesh $\mathbf{F} = \{\mathbf{V}_1, \mathbf{V}_6, \mathbf{V}_5, \mathbf{V}_4\}$

$$\mathbf{L}(u,v) = (1-v)[(1-u)\mathbf{V}_1 + u\mathbf{V}_6] + v[(1-u)\mathbf{V}_4 + u\mathbf{V}_5], \qquad 0 \le u, v \le 1$$

then the maximum distance between $\mathbf{S}(u,v)$ and its control mesh can be written as

$$\| \, \mathbf{L}(u,v) - \mathbf{S}(u,v) \, \| \ \le \| \, \mathbf{L}(u,v) - \mathbf{L}_b^m(u_m, v_m) \| + \| \mathbf{L}_b^m(u_m, v_m) - \mathbf{S}(u,v) \, \| \tag{8}$$

where $0 \le u, v \le 1$ and u_m and v_m are defined in (6). The second term on the right hand side of the inequality can be evaluated using (7). Hence, one only needs to work with the first term on the right hand side of the inequality.

It is easy to see that if $(u,v) \in \Omega_b^m$ then $(u,v) \in \Omega_0^k$ for any $0 \le k < m$ where

$$\Omega_0^k = [0, \ \frac{1}{2^k}] \times [0, \ \frac{1}{2^k}].$$

Ω_0^k corresponds to the subpatch $\mathbf{S}_0^k$. This means that $(2^k u, 2^k v)$ is within the parameter space of $\mathbf{S}_0^k$ for $0 \le k < m$, i.e., $(2^k u, 2^k v) = (u_k, v_k)$ where u_k and v_k are defined in (6). Consequently, we can consider $\mathbf{L}_0^k(u_k, v_k)$ for $0 \le k < m$ where $\mathbf{L}_0^k$ is the bilinear parametrization of the center face of the control mesh of $\mathbf{S}_0^k$ (with the understanding that $\mathbf{L}_0^0 = \mathbf{L}$ and $(u_0, v_0) = (u,v)$). Hence, the first term on the right hand side of (8) can be written as

$$\|\mathbf{L}(u,v) - \mathbf{L}_b^m(u_m,v_m)\| \le \sum_{k=0}^{m-2} \|\mathbf{L}_0^k(u_k,v_k) - \mathbf{L}_0^{k+1}(u_{k+1},v_{k+1})\| + \|\mathbf{L}_0^{m-1}(u_{m-1},v_{m-1}) - \mathbf{L}_b^m(u_m,v_m)\|. \tag{9}$$

The following two lemmas are needed in the evaluation of the right side of the above inequality.

Lemma 3: If $(u,v) \in \Omega_b^m$ where b and m are defined in (5) then for any $0 \le k < m-1$ we have

$$\| \, \mathbf{L}_0^k(u_k,v_k) - \mathbf{L}_0^{k+1}(u_{k+1},v_{k+1}) \, \| \le \frac{1}{\min\{ \, n, 8 \, \}} M_k$$

where M_k is the second order norm of $\mathbf{M}_k$ and $\mathbf{L}_0^0 = \mathbf{L}$.

Lemma 4: If $(u,v) \in \Omega_b^m$ where b and m are defined in (5) then we have

$$\| \, \mathbf{L}_0^{m-1}(u_{m-1},v_{m-1}) - \mathbf{L}_b^m(u_m,v_m) \, \| \le \begin{cases} \frac{1}{4} M_{m-1}, & if \quad b = 2 \\ \frac{1}{8} M_{m-1}, & if \quad b = 1 \ or \ 3 \end{cases}$$

where M_{m-1} is the second order norm of $\mathbf{M}_{m-1}$.

By applying Lemmas 3 and 4 on (9) and then using (7) on (8), we have the following lemma on the distance between an extra-ordinary CCSS patch $\mathbf{S}(u,v)$ and its control mesh $\mathbf{L}(u,v)$ [5].

Lemma 5: The maximum of $\| \, \mathbf{L}(u,v) - \mathbf{S}(u,v) \, \|$ satisfies the following inequality

$$\| \, \mathbf{L}(u,v) - \mathbf{S}(u,v) \, \| \le \begin{cases} M_0, & n = 3 \\ \frac{5}{7} M_0, & n = 5 \\ \frac{4n}{n^2-8n+46} M_0, & 5 < n \le 8 \\ \frac{n^2}{4(n^2-8n+46)} M_0, & n > 8 \end{cases} \tag{10}$$

where $M = M_0$ is the second order norm of the extra-ordinary patch $\mathbf{S}(u,v)$.

2.2.2 Subdivision Depth Computation

Lemma 5 can be used to estimate the distance between a level-k control mesh and the surface patch for any $k > 0$. This is because the distance between a level-k control mesh and the surface patch is dominated by the distance between the level-k extra-ordinary subpatch and the corresponding control mesh which, according to Lemma 5, is

$$\| \mathbf{L}_k(u,v) - \mathbf{S}(u,v) \| \le \begin{cases} M_k, & n = 3 \\ \frac{18}{25} M_k, & 5 \le n \le 8 \\ \frac{n^2}{4(n^2-8n+46)} M_k, & n > 8 \end{cases}$$

where M_k is the second order norm of $\mathbf{S}(u,v)$'s level-k control mesh $\mathbf{M}_k$. The previous subdivision depth computation technique for extra-ordinary surface patches is obtained by combining the above result with Lemma 2 [5].

Theorem 6: Given an extra-ordinary surface patch $\mathbf{S}(u,v)$ and an error tolerance ϵ, if k levels of subdivisions are iteratively performed on the control mesh of $\mathbf{S}(u,v)$, where

$$k = \left\lceil log_w \frac{M}{z\epsilon} \right\rceil$$

with M being the second order norm of $\mathbf{S}(u,v)$ defined in (3),

$$w = \begin{cases} \frac{3}{2}, & n = 3 \\ \frac{25}{18}, & n = 5 \\ \frac{4n^2}{3n^2+8n-46}, & n > 5 \end{cases} \quad and \quad z = \begin{cases} 1, & n = 3 \\ \frac{25}{18}, & 5 \le n \le 8 \\ \frac{2(n^2-8n+46)}{n^2}, & n > 8 \end{cases}$$

then the distance between $\mathbf{S}(u,v)$ and the level-k control mesh is smaller than ϵ.

3 New Subdivision Depth Computation Technique for Extra-Ordinary Patches

The SOFDs involved in the second order norm of an extra-ordinary CCSS patch (see eq. (3)) can be classified into two groups: *group I* and *group II*. Group I contains those SOFDs that involve vertices in the vicinity of the extra-ordinary vertex (see Figure 6(a)). These are the first $2n$ SOFDs in (3). Group II contains the remaining SOFDs, i.e., SOFDs that involve vertices in the vicinity of the other three vertices of $\mathbf{S}$ (see Figure 6(b)). These are the last 10 SOFDs in (3). It is easy to see that the convergence rate of the SOFDs in group II is the same as the regular case, i.e., 1/4 [4]. Therefore, to study properties of the second order norm M, it is sufficient to study norms of the SOFDs in group I. The maximum of these norms will be called the *second order norm* of group I. We will use $M = M_0$ to represent group I's second order norm as well because norms of group I's SOFDs dominate norms of group II's SOFDs. For convenience of reference, in the subsequent discussion we shall simply use the term "second order norm of an extra-ordinary CCSS patch" to refer to the "second order norm of group I of an extra-ordinary CCSS patch".

3.1 Matrix based Rate of Convergence

The second order norm of $\mathbf{S} = \mathbf{S}_0^0$ can be put in matrix form as follows:

$$M = \|\mathrm{A}\mathbf{P}\|_\infty$$

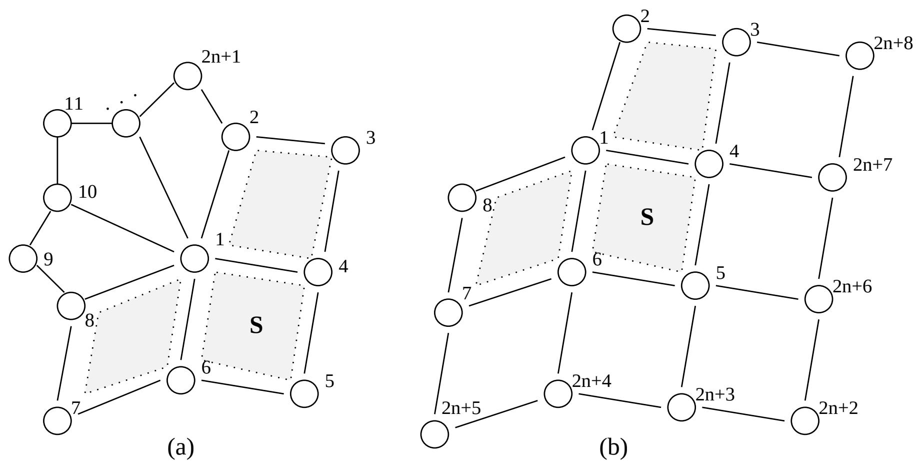

Figure 6: (a) Vicinity of the extra-ordinary point. (b) Vicinity of the other three vertices of $\mathbf{S}$.

where A is a $2n * (2n+1)$ matrix

$$\mathrm{A} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 2 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & \cdots & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ & & & & \vdots & & & & & & \\ 2 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & \cdots & -1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ & & & & \vdots & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 2 & -1 \end{bmatrix}$$

and $\mathbf{P}$ is a control point vector

$$\mathbf{P} = [\mathbf{V}_1,\ \mathbf{V}_2,\ \mathbf{V}_3, \ldots,\ \mathbf{V}_{2n+1}]^T.$$

A is called the *second order norm matrix* for extra-ordinary CCSS patches. If i levels of Catmull-Clark subdivision are performed on the control mesh of $\mathbf{S} = \mathbf{S}_0^0$ then, following the notation of Section 2, we have an extra-ordinary subpatch $\mathbf{S}_0^i$ whose second order norm can be expressed as:

$$M_i = \left\| \mathrm{A}\Lambda^i \mathbf{P} \right\|_\infty$$

where Λ is a subdivision matrix of dimension $(2n+1) * (2n+1)$. The function of Λ is to perform a subdivision step on the $2n+1$ control vertices around (and including) the extra-ordinary point (see Figure 6(a)). For example, when $n = 3$, Λ is of the following form:

$$\Lambda = \begin{bmatrix} 5/12 & 1/6 & 1/36 & 1/6 & 1/36 & 1/6 & 1/36 \\ 3/8 & 3/8 & 1/16 & 1/16 & 0 & 1/16 & 1/16 \\ 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 & 0 \\ 3/8 & 1/16 & 1/16 & 3/8 & 1/16 & 1/16 & 0 \\ 1/4 & 0 & 0 & 1/4 & 1/4 & 1/4 & 0 \\ 3/8 & 1/16 & 0 & 1/16 & 1/16 & 3/8 & 1/16 \\ 1/4 & 1/4 & 0 & 0 & 0 & 1/4 & 1/4 \end{bmatrix}.$$

We are interested in knowing the relationship between $\|\mathrm{A}\mathbf{P}\|_\infty$ and $\left\|\mathrm{A}\Lambda^i\mathbf{P}\right\|_\infty$. We need two lemmas for this relationship. The first one shows the explicit form of $\mathrm{A}^+\mathrm{A}$ where A^+ is the pseudo-inverse of A. The second one shows that A^+A can act as a right identity matrix for $A\Lambda^i$.

Lemma 7: The product of the second order norm matrix A and its pseudo-inverse matrix A^+ can be expressed as follows:

$$A^+A = \begin{cases} H, & n = 2k+1 \\ H + E, & n = 4k+2 \\ H + E + W + Z, & n = 4k \end{cases} \quad (11)$$

where k is a positive integer, and H, E, W and Z are $(2n+1) * (2n+1)$ matrices of the following form with H being a circulant matrix:

$$H \equiv \frac{1}{2n+1} \begin{bmatrix} 2n & -1 & \cdots & -1 & -1 \\ -1 & 2n & \cdots & -1 & -1 \\ & \vdots & & \vdots & \\ -1 & -1 & \cdots & 2n & -1 \\ -1 & -1 & \cdots & -1 & 2n \end{bmatrix}, \qquad E = \frac{1}{n} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & \cdots & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ & & \vdots & & \vdots & & & \\ 0 & 0 & -1 & 0 & 1 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & \cdots & -1 \end{bmatrix},$$

$$W = \frac{2}{3n} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & -1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & -1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 0 & 1 & 0 & \cdots & 0 \\ & \vdots & & \vdots & & & \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 0 & 1 & 0 & \cdots & 0 \end{bmatrix}, \qquad Z = \frac{2}{3n} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & \cdots & 1 \\ 0 & 0 & -2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & \cdots & -1 \\ 0 & 0 & 0 & 0 & -2 & 0 & \cdots & -2 \\ 0 & 0 & 1 & 0 & -1 & 0 & \cdots & -1 \\ 0 & 0 & 2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 & \cdots & 2 \\ & & \vdots & & \vdots & & & \\ 0 & 0 & 1 & 0 & 1 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 & \cdots & 2 \end{bmatrix}.$$

Proof We prove that if $n = 2k+1$ for some positive integer k then $A^+A = H$ where H is defined above. From properties of pseudo-inverse matrices [2], we know that

$$A^+A = A^LA$$

where A^L is a *left weak generalized inverse matrix* of A, i.e., A^L is a matrix satisfying the following conditions

$$\begin{aligned} AA^LA &= A \\ A^LAA^L &= A^L \\ (A^LA)^T &= A^LA \end{aligned} \quad (12)$$

Thus, to prove $A^+A = H$, we just need to show that there exists a left weak generalized matrix A^L of A such that $A^LA = H$. We first prove that there exists a $(2n+1) * (2n)$ matrix C such that

$$C\,A = H. \quad (13)$$

(13) is equivalant to

$$\mathrm{A}^T\mathrm{C}^T = \mathrm{A}^T\left[\mathbf{C}_1\ \mathbf{C}_2\ \cdots\ \mathbf{C}_{2n+1}\right] = \mathrm{H}^T = \mathrm{H} = \left[\mathbf{H}_1\ \mathbf{H}_2\ \cdots\ \mathbf{H}_{2n+1}\right]$$

where $\mathbf{C}_i^T$ are row vectors of C and $\mathbf{H}_i$ are column vectors of H. This is a system of $2n+1$ linear equations: $\mathrm{A}^T\mathbf{C}_i = \mathbf{H}_i$, $i = 1, 2, ..., 2n+1$. Each of these systems has a solution $\mathbf{C}_i$ because

$$rank(\mathrm{A}^T) = rank(\overline{\mathrm{A}_i^T}) < 2n+1$$

where $\overline{\mathrm{A}_i^T} = \left[\mathrm{A}^T\ \mathbf{H}_i\right]$. Hence, there is at least one solution for C in (13) when $n = 2k+1$.

It can be proved that there is no solution for $\mathrm{CA} = \mathrm{H}$ when $n = 4k+2$ because for some $\mathbf{C}_i$ we would have $rank(\mathrm{A}^T) < rank(\overline{\mathrm{A}_i^T})$. However, there is at least one solution for $\mathrm{CA} = \mathrm{H} + \mathrm{E}$. Same for $\mathrm{CA} = \mathrm{H} + \mathrm{E} + \mathrm{W} + \mathrm{Z}$ when $n = 4k$.

It is easy to verify that, when $n = 2k+1$, the matrix C satisfies conditions 1 and 3 in (12), i.e.,

$$\mathrm{ACA} = \mathrm{AH} = \mathrm{A} \qquad and \qquad (\mathrm{CA})^T = \mathrm{CA}.$$

As far as the second condition is concerned, there are two possibilities for CAC:

Case 1: $\mathrm{CAC} = \mathrm{C}$

In this case, C is a left weak generalized inverse of matrix A. Hence, we have $\mathrm{A}^+\mathrm{A} = \mathrm{CA} = \mathrm{H}$.

Case 2: $\mathrm{CAC} = \mathrm{C} + \mathrm{D}$, where $\mathrm{D} \neq 0$.

We claim, in this case, $\mathrm{C}+\mathrm{D}$ is a left weak generalized matrix of A and $\mathrm{C}+\mathrm{D}$ is also a solution of (13). We first show that $\mathrm{C}+\mathrm{D}$ is also a solution of (13). Note that $\mathrm{H}^2 = \mathrm{H}$. Hence, we have:

$$(\mathrm{C}+\mathrm{D})\mathrm{A} = \mathrm{CACA} = \mathrm{H}^2 = \mathrm{H} = \mathrm{CA}.$$

This also shows that $\mathrm{DA} = 0$. To prove that $\mathrm{C}+\mathrm{D}$ is a left weak generalized matrix of A, note that

$$\begin{aligned} &\mathrm{A}(\mathrm{C}+\mathrm{D})\mathrm{A} = \mathrm{ACA} + \mathrm{ADA} = \mathrm{ACA} = \mathrm{A}, \qquad and \\ &(\mathrm{C}+\mathrm{D})\mathrm{A}(\mathrm{C}+\mathrm{D}) = \mathrm{CA}(\mathrm{C}+\mathrm{D}) + \mathrm{DA}(\mathrm{C}+\mathrm{D}) = \mathrm{CA}(\mathrm{C}+\mathrm{D}) \\ &\qquad = \mathrm{CAC} + \mathrm{CAD} = \mathrm{CAC} = \mathrm{C}+\mathrm{D} \end{aligned}$$

The second equation is true because

$$\mathrm{CAC} = \mathrm{CACAC} = \mathrm{CA}(\mathrm{C}+\mathrm{D}) = \mathrm{CAC} + \mathrm{CAD}.$$

Therefore, the first and second conditions of (12) are satisfied. We also have $((\mathrm{C}+\mathrm{D})\mathrm{A})^T = (\mathrm{C}+\mathrm{D})\mathrm{A}$ because $(\mathrm{C}+\mathrm{D})\mathrm{A} = \mathrm{H}$ and H is a symmetric matrix. Hence, $\mathrm{C}+\mathrm{D}$ is indeed a left weak generalized matrix of A. Consequently, we have $\mathrm{A}^+\mathrm{A} = (\mathrm{C}+\mathrm{D})\mathrm{A} = \mathrm{H}$.

The other two cases $n = 4k+2$ and $n = 4k$ can be proved similarly. □

<u>Lemma 8:</u> $\mathrm{A}^+\mathrm{A}$ is a right identity matrix of $\mathrm{A}\Lambda^i$, i.e., $\mathrm{A}\Lambda^i\mathrm{A}^+\mathrm{A} = \mathrm{A}\Lambda^i$, for any i.

<u>Proof</u> We prove the case $n = 2k+1$ first. Let F be a $(2n+1)*(2n+1)$ Fourier transform matrix

$$\mathrm{F} = \frac{1}{\sqrt{2n+1}}\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{2n-1} & \omega^{2n} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{4n-2} & \omega^{4n} \\ \vdots & & & \vdots & & \vdots \\ 1 & \omega^{2n} & \omega^{4n} & \cdots & \omega^{4n^2-2n} & \omega^{4n^2} \end{bmatrix}$$

where $\omega = e^{2\pi i/(2n+1)}$. It is easy to see that

$$\mathrm{F}^*\mathrm{HF} = \mathrm{I} - \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

where I is a $(2n+1) * (2n+1)$ identity matrix. Hence, when $n = 2k+1$ we have

$$\mathrm{A}\Lambda^i\mathrm{A}^+\mathrm{A} = \mathrm{A}\Lambda^i\mathrm{H} = \mathrm{A}\Lambda^i\mathrm{FF}^*\mathrm{HFF}^* = \mathrm{A}\Lambda^i\mathrm{F}(\mathrm{I} - \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix})\mathrm{F}^*$$

$$= \mathrm{A}\Lambda^i - \mathrm{A}\Lambda^i\mathrm{F} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \mathrm{F}^* = \mathrm{A}\Lambda^i - \mathrm{A}\Lambda^i \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & & & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

Note that

$$\mathrm{A}\Lambda^i \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & & & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} = 0$$

because the row sum of A is 0 and row sum of Λ is 1. Hence, we have $\mathrm{A}\Lambda^i = \mathrm{A}\Lambda^i\mathrm{A}^+\mathrm{A}$ when $n = 2k+1$.

We next prove the lemma for $n = 4k+2$. Note that in this case $\Lambda\mathrm{E} = \frac{1}{4}\mathrm{E}$ and $\mathrm{AE} = 0$. With these results we have

$$\mathrm{A}\Lambda^i\mathrm{E} = \frac{1}{4^i}\mathrm{AE} = 0.$$

Hence, $\mathrm{A}\Lambda^i\mathrm{A}^+\mathrm{A} = \mathrm{A}\Lambda^i(\mathrm{H}+\mathrm{E}) = \mathrm{A}\Lambda^i$.

Finally, we prove the lemma for $n = 4k$. Similar to the previous case, we can prove that $\Lambda\mathrm{W} = \frac{1}{2}\mathrm{W}$, $\mathrm{AW} = 0$ and $\Lambda\mathrm{Z} = \frac{1}{2}\mathrm{Z}_1$, $\mathrm{AZ} = 0$. Therefore, we have $\mathrm{A}\Lambda^i\mathrm{W} = \frac{1}{2^i}\mathrm{AW} = 0$ and $\mathrm{A}\Lambda^i\mathrm{Z} = \frac{1}{2^i}\mathrm{AZ} = 0$. Hence, $\mathrm{A}\Lambda^i\mathrm{A}^+\mathrm{A} = \mathrm{A}\Lambda^i(\mathrm{H}+\mathrm{E}+\mathrm{W}+\mathrm{Z}) = \mathrm{A}\Lambda^i$. $\square$

With this lemma, we have

$$\frac{\|\mathrm{A}\Lambda^i\mathbf{P}\|_\infty}{\|\mathrm{A}\mathbf{P}\|_\infty} = \frac{\|\mathrm{A}\Lambda^i\mathrm{A}^+\mathrm{A}\mathbf{P}\|_\infty}{\|\mathrm{A}\mathbf{P}\|_\infty} \le \frac{\|\mathrm{A}\Lambda^i\mathrm{A}^+\|_\infty \|\mathrm{A}\mathbf{P}\|_\infty}{\|\mathrm{A}\mathbf{P}\|_\infty} = \left\|\mathrm{A}\Lambda^i\mathrm{A}^+\right\|_\infty$$

Use r_i to represent $\|\mathrm{A}\Lambda^i\mathrm{A}^+\|_\infty$. Then, for any $0 < j < i$, we have the following recurrence formula for r_i

$$r_i \equiv \left\|\mathrm{A}\Lambda^i\mathrm{A}^+\right\|_\infty = \left\|\mathrm{A}\Lambda^{i-j}\mathrm{A}^+\mathrm{A}\Lambda^j\mathrm{A}^+\right\|_\infty \le \left\|\mathrm{A}\Lambda^{i-j}\mathrm{A}^+\right\|_\infty \left\|\mathrm{A}\Lambda^j\mathrm{A}^+\right\|_\infty = r_{i-j}\ r_j \tag{14}$$

where $r_0 = 1$. Hence, we have the following lemma on the convergence rate of second order norm of an extra-ordinary CCSS patch.

Lemma 9: The second order norm of an extra-ordinary CCSS patch satisfies the following inquality:

$$M_i \le r_i\ M_0 \tag{15}$$

where $r_i = \|A\Lambda^i A^+\|_\infty$ and r_i satisfies the recurrence formula (14).

The recurrence formula (14) shows that r_i in (15) can be replaced with r_1^i. However, experiment data show that, while the convergence rate changes by a constant ratio in most of the cases, there is a significant difference between r_2 and r_1. The value of r_2 is smaller than r_1^2 by a significant gap. Hence, if we use r_1^i for r_i in (15), we would end up with a bigger subdivision depth for a given error tolerance. A better choice is to use r_2 to bound r_i, as follows.

$$r_i \leq \begin{cases} r_2^j, & \mathrm{i} = 2j \\ r_1 r_2^j, & \mathrm{i} = 2j+1 \end{cases} \tag{16}$$

3.2 Distance Evaluation

Following (8) and (9), the distance between the extra-ordinary CCSS patch $\mathbf{S}(u,v)$ and the center face of its control mesh $\mathbf{L}(u,v)$ can be expressed as

$$\begin{aligned}\|\mathbf{L}(u,v) - \mathbf{S}(u,v)\| \leq \textstyle\sum_{k=0}^{m-2} \|\mathbf{L}_0^k(u_k,v_k) - \mathbf{L}_0^{k+1}(u_{k+1},v_{k+1})\| + \|\mathbf{L}_0^{m-1}(u_{m-1},v_{m-1}) - \mathbf{L}_b^m(u_m,v_m)\| \\ + \|\mathbf{L}_b^m(u_m,v_m) - \mathbf{S}_b^m(u_m,v_m)\|\end{aligned} \tag{17}$$

where m and b are defined in (5) and (u_i, v_i) are defined in (6). By applying Lemma 3, Lemma 4 and (7) on the first, second and third terms of the right hand side of the above inequality, respectively, we get

$$\|\mathbf{L}(u,v) - \mathbf{S}(u,v)\| \leq c\sum_{k=0}^{m-2} M_k + \frac{1}{4}M_{m-1} + \frac{1}{3}M_m \leq M_0(c\sum_{k=0}^{m-2} r_k + \frac{1}{4}r_{m-1} + \frac{1}{3}r_m)$$

where $c = 1/\min\{n,8\}$. The last part of the above inequality follows from Lemma 8. Consequently, through a simple algebra, we have

$$\|\mathbf{L}(u,v) - \mathbf{S}(u,v)\| \leq \begin{cases} M_0[c(\frac{1-r_2^j}{1-r_2} + \frac{1-r_2^{j-1}}{1-r_2}r_1) + \frac{r_1 r_2^{j-1}}{4} + \frac{r_2^j}{3}], & \text{if} \quad m = 2j \\ M_0[c(\frac{1-r_2^j}{1-r_2} + \frac{1-r_2^j}{1-r_2}r_1) + \frac{r_2^j}{4} + \frac{r_1 r_2^j}{3}], & \text{if} \quad m = 2j+1 \end{cases}$$

It can be easily proved that the maximum occurs at $m = \infty$. Hence, we have the following lemma.

<u>**Lemma 10:**</u> The maximum of $\|\mathbf{L}(u,v) - \mathbf{S}(u,v)\|$ satisfies the following inequality

$$\|\mathbf{L}(u,v) - \mathbf{S}(u,v)\| \leq \frac{M_0}{\min\{n,8\}} \frac{1+r_1}{1-r_2}$$

where $r_i = \|A\Lambda^i A^+\|_\infty$ and $M = M_0$ is the second order norm of the extra-ordinary patch $\mathbf{S}(u,v)$.

3.3 Subdivision Depth Computation

Lemma 9 can also be used to evaluate the distance between a level-i control mesh and the extra-ordinary patch $\mathbf{S}(u,v)$ for any $i > 0$. This is because the distance between a level-i control mesh and the surface patch $\mathbf{S}(u,v)$ is dominated by the distance between the level-i extra-ordinary subpatch and the corresponding control mesh which, accoriding to Lemma 9, is

$$\|\mathbf{L}_i(u,v) - \mathbf{S}(u,v)\| \leq \frac{M_i}{\min\{n,8\}} \frac{1+r_1}{1-r_2}$$

where M_i is the second order norm of $\mathbf{S}(u,v)$'s level-i control mesh, $\mathbf{M}_i$. Hence, if the right side of the above inequality is smaller than a given error tolerance ϵ, then the distance between $\mathbf{S}(u,v)$ and the level-i control mesh is smaller than ϵ. Consequently, we have the following subdivision depth computation theorem for extra-ordinary CCSS patches.

Theorem 11: Given an extra-ordinary surface patch $\mathbf{S}(u,v)$ and an error tolerance ϵ, if

$$i \equiv \min\{2l,\ 2k+1\}$$

levels of subdivision are iteratively performed on the control mesh of $\mathbf{S}(u,v)$, where

$$l = \lceil \log_{\frac{1}{r_2}} \left(\frac{1}{\min\{n,8\}} \frac{1+r_1}{1-r_2} \frac{M_0}{\epsilon} \right) \rceil \ , \qquad k = \lceil \log_{\frac{1}{r_2}} \left(\frac{r_1}{\min\{n,8\}} \frac{1+r_1}{1-r_2} \frac{M_0}{\epsilon} \right) \rceil$$

with $r_i = \|\mathrm{A}\Lambda^i\mathrm{A}^+\|_\infty$ and M_0 being the second order norm of $\mathbf{S}(u,v)$, then the distance between $\mathbf{S}(u,v)$ and the level-i control mesh is smaller than ϵ.

4 Examples

The new subdivision depth technique has been inplemented in *C++* on the Windows platform to compare its performance with the previous approach. *MatLab* is used for both numerical and symbolic computation of r_i in the implementation. Table 1 shows the comparison results of the previous technique, Theorem 6, with the new technique, Theorem 10. Two error tolerances 0.01 and 0.001 are considered and the second order norm M_0 is assumed to be 2. For each error tolerance, we consider five different valences: 3, 5, 6, 7 and 8 for the extra-ordinary vertex. As can be seen from the table, the new technique has a 30% improvement over the previous technique in most of the cases. Hence, the new technique indeed improves the previous technique significantly.

To show that the rates of convergence are indeed difference between r_1 and r_2, their values from several typical extra-ordinary CCSS patches are included in Table 2. Note that when we compare r_1 and r_2, the value of r_1 should be squared first.

Table 1. Comparison between the old technique and the new technique

	$\epsilon = 0.01$		$\epsilon = 0.001$	
N	Old Technique	New Technique	Old Technique	New Technique
3	14	9	19	12
5	16	11	23	16
6	19	16	27	22
7	23	14	33	22
8	37	27	49	33

Table 2. Values of r_1 and r_2 for some extra-ordinary patches.

N	r_1	r_2
3	0.6667	0.2917
5	0.7200	0.4016
6	0.8889	0.5098
7	0.8010	0.5121
8	1.0078	0.5691

5 Conclusions

A new subdivision depth computation technique for extra-ordinary CCSS patches is presented. Like the previous technique, the subdivision depth is computed based on norms of the second order forward differences of the control points. However, the computation process is performed on matrix representation of the second order norm, which gives us a better bound of the convergence rate and, consequently, a tighter subdivision depth for a given error tolerance. Test results show that the new technique improves the previous technique by about 30% in most of the cases. This is a significant result because of the exponential nature of the subdivision process. We are not sure if the new technique can be further improved though.

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