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Beta-Bezier Curves

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Abstract. A new definition of Beta-Bezier curves which include classic Bezier curves as a special case is given. With the new definition, the functions of Beta-Bezier curves are easier to study. It shows that Beta-Bezier curves not only have all the basic properties of Bezier curves such as convex hull property, recursive subdivision, B-spline conversion and C^2 interpolation, but also the capability of modifying the shape a Bezier curve segment or a C^2-continuous, composite cubic Bezier curve without changing the control points of the curve. This is because in the cubic case a Beta-Bezier curve is actually also a Bezier curve. Hence, we have a curve design technique more general than Bezier curves. Since C^2-continuous, composite cubic Bezier curves are equivalent to uniform B-spline curves, this means the new curve design technique is more general than uniform B-spline curves as well.

Keywords: Bezier curve, Beta-Bernstein basis function, shape parameter, Beta-Bezier curve.

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1 INTRODUCTION

The shape of a Bezier curve segment is guided by its control polygon. By manipulating the control points of a Bezier curve segment, one can manipulate the shape of the control polygon and, consequently, the shape of the curve segment. In addition, Bezier curve segments satisfy the convex hull property [4; 5; 8] and subdivision property [4; 5; 8] which make clipping, rendering and intersection computation for Bezier curve segments more efficient or doable. Hence, Bezier curves not only have intuitive appeal for interactive users, but are also attractive for system and numerical applications.

Actually cubic Bezier curves are popular for shape modeling applications, such as font, cartoon character and car body design/representation (Figure 1), as well. Given a set of data points, one can construct an open or closed, composite cubic Bezier curve that interpolates the data points at its joints [2]. The composite curve is C^2-continuous and can be manipulated by manipulating the data points. The final curve can be converted into a cubic B-spline curve [2] and, hence, can be processed either as a cubic B-spline curve or a composite cubic Bezier curve.

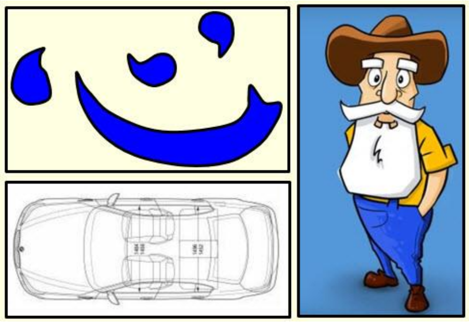

Figure 1. 2D shape modeling examples.

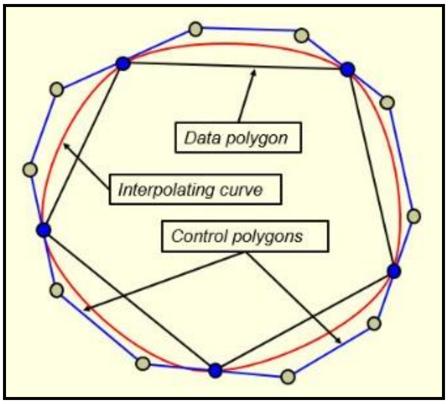

Figure 2. A composite cubic Bezier curve that interpolates a set of data points.

While one can change the skeleton shape (shape of the data polygon, see Figure 2) of the interpolating composite cubic Bezier curve (ICCBC), one cannot change the shape of the ICCBC once the skeleton shape is determined. This is because the control polygons of the segments of the ICCBC are completely determined by the shape of the data polygon. Hence, once the shape of the data polygon is determined, then so is the shape of the ICCBC.

For years people have been trying to find ways to extend/modify the definition of Bezier curves so that one can change the shape of the curve without changing the control points of the curve [1; 6; 9-11]. But none of the works seem to be intuitive enough for practical applications in the field. A recent work by Chu and Zeng [3] was an attempt in that direction as well. A Beta-Bezier curve of degree n with shape parameter $\lambda \geq 0$ is defined by

$$C(t;\lambda) = \sum_{k=0}^{n} P_k \beta_k^n(t;\lambda), \qquad 0 \leq t \leq 1$$

where

$$\beta_k^n(t;\lambda) = \binom{n}{k}\frac{\prod_{i=0}^{k-1}(\lambda t+i)\prod_{j=0}^{n-1-k}[\lambda(1-t)+j]}{\prod_{m=0}^{n-1}(\lambda+m)}, \tag{1}$$

are the so-called Beta-Bernstein basis functions, P_k are 2D or 3D control points.

When λ tends to infinity, Beta-Bezier basis functions reduce to Bernstein basis functions

$$B_{n,k}(t) = \binom{n}{k}(1-t)^{n-k}t^k$$

Hence the Beta-Bezier curves defined by Chu and Zeng [3] include Bezier curves as a special case. A few geometric properties of the curve, including a de Casteljau like algorithm similar to Bezier curve's de Casteljau algorithm, are studied [3]. Conditions on C^1-continuity at the joint of two adjacent Beta-Bezier curve segments are discussed by Levent and Sahin [7]. Unfortunately, Chu/Zeng and Levent/Sahin did not realize that the definition of Beta-Bezier curves given in [3] is not the best definition for Beta-Bezier curves.

In this paper, we will present a better definition for Beta-Bezier curves and show that, with this new definition, properties of Beta-Bezier curves can be easily studied and computed, such as showing that Beta-Bezier curves satisfy the convex hull property and computing the second derivative of a cubic Beta-Bezier curve. Consequently, we are not only able to modify the shape of a Beta-Bezier curve without changing the control points of the curve, but also to perform all the properties of a Bezier curve such as recursive subdivision, converting to a B-spline representation, joining two curve segments with C^2-smoothness and interpolating a set of data points with a composite cubic Beta-Bezier curves that is C^2-continuous. One of the reasons for us to be able to do these is that in the cubic case, a Beta-Bezier curve is actually also a Bezier curve.

The rest of the paper is arranged as follows. In section 2, a new definition of Beta-Bezier curves is presented and basic properties of Beta-Bezier curves defined this way are studied. Further properties of Beta-Bezier curves such as smooth (C^2-) joining of two curve segments, subdivision property, C^2-continuous interpolation and B-spline conversion are discussed in Sections 3, 4, 5 and 6, respectively. Concluding remarks are given in Section 7.

2 NEW DEFINITION OF BETA-BEZIER CURVES AND BASIC PROPERTIES

A Beta-Bezier curve of degree n with shape parameter $\beta \geq 0$ is defined as follows

$$\mathrm{C}(t;\beta) = \sum_{k=0}^{n} P_k B_{n,k}(t;\beta), \qquad 0 \leq t \leq 1 \tag{2}$$

where $P_0, P_1, \ldots, P_n$ are 2D or 3D control points and

$$B_{n,k}(t;\beta) = \frac{\binom{n}{k}\prod_{i=0}^{n-1-k}(1-t+i\beta)\prod_{j=0}^{k-1}(t+j\beta)}{\prod_{m=0}^{n-1}(1+m\beta)}, \tag{3}$$

$k = 0, 1, \ldots, n$, are Beta-Bernstein basis functions of degree n. The basis functions defined in (3) are related to the basis functions defined in (1) in that $\beta = 1/\lambda$. We have the following immediate properties of Beta-Bézier curves:

(i) When $\beta = 0$, $B_{n,k}(t;0) = \binom{n}{k}(1-t)^{n-k}t^k$. Hence $C(t;\beta)$ reduces to a Bezier curve $C(t)$ of degree n defined as follows when $\beta = 0$

$$C(t) = \sum_{k=0}^{n} P_k B_{n,k}(t), \qquad 0 \leq t \leq 1 \tag{4}$$

where

$$B_{n,k}(t) = \binom{n}{k}(1-t)^{n-k}t^k \tag{5}$$

are Bernstein basis functions of degree *n*.

(ii) Since $B_{n,0}(0;\beta) = 1$ and $B_{n,k}(0;\beta) = 0$ for all k bigger than 0, and $B_{n,n}(1;\beta) = 1$ and $B_{n,k}(1;\beta) = 0$ for all k smaller than n, a Beta-Bezier curve segment always starts at the first control point P_0 and ends at the last control point P_n.

(iii) The sum of the basis functions of a Beta-Bezier curve equals one for any t and β （the unit sum property）. Hence Beta-Bezier curves also satisfy the "convex hull property", that is, a Beta-Bezier curve segment is always contained in the convex hull of its control points. A proof to show that the basis functions of a Beta-Bezier curve satisfy the unit sum property is given in Appendix A.

(iv) A non-zero β applies a dripping force to the curve. The bigger the value of β, the bigger the dripping force. The dripping force pulls the curve segment towards the base line segment $\overline{P_0P_n}$ of the curve. When $\beta = +\infty$, the curve coincides with the base line segment. See Figure 3 for the cases when $\beta = 0$, $\beta = 1$ and $\beta = +\infty$ for a cubic Beta-Bezier curve segment.

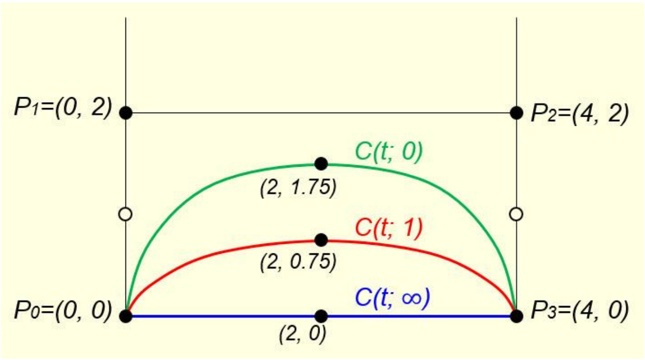

Figure 3. Cubic Beta-Bezier curves defined by the same control points but with different shape parameters.

From this point on we will mainly focus on degree 3 case because that is what people are using for most of the applications. The only exception we know is a degree five curve/surface used in a dam design in China. But it is known that dam design case can be done using degree three curve/surface as well.

(v) A cubic Beta-Bezier curve can be represented as a cubic Bezier curve.

Given a cubic Beta-Bezier curve $C(t;\beta)$ with control points P_0, P_1, P_2, P_3, defined as follows

$$C(t;\beta) = \sum_{k=0}^{3} B_{3,k}(t;\beta)P_k, \tag{6}$$

where $B_{3,k}(t;\beta)$ are Beta-Bernstein basis functions of degree 3 as defined in (3), through simple computation, one can rewrite $C(t;\beta)$ as a cubic Bezier curve as follows:

$$C(t;\beta) = (1-t)^3P_0 + 3t(1-t)^2Q_1 + 3t^2(1-t)Q_2 + t^3P_3, \tag{7}$$

where

$$Q_1 = \frac{\beta(3+4\beta)}{3(1+\beta)(1+2\beta)}P_0 + \frac{1}{1+2\beta}P_1 + \frac{\beta}{(1+\beta)(1+2\beta)}P_2 + \frac{2\beta^2}{3(1+\beta)(1+2\beta)}P_3, \tag{8}$$

$$Q_2 = \frac{2\beta^2}{3(1+\beta)(1+2\beta)}P_0 + \frac{\beta}{(1+\beta)(1+2\beta)}P_1 + \frac{1}{1+2\beta}P_2 + \frac{\beta(3+4\beta)}{3(1+\beta)(1+2\beta)}P_3 \tag{9}$$

The sums of the coefficients of P_0, P_1, P_2 and P_3 in Q_1 and Q_2 are both 1. So Q_1 and Q_2 are both inside the convex hull of the control points of $C(t;\beta)$. Since the sum of the coefficients of P_0, Q_1, Q_2

and P_3 in $C(t;\beta)$ is one, this provides another proof of the fact that cubic Beta-Bezier curves satisfy the "convex hull property".

Q_1 and Q_2 can be written as

$$Q_1 = \frac{1}{1+\beta}\left(\frac{1+\beta}{1+2\beta}P_1 + \frac{\beta}{1+2\beta}P_2\right) + \frac{\beta}{1+\beta}\left(\frac{1+4\beta/3}{1+2\beta}P_0 + \frac{2\beta/3}{1+2\beta}P_3\right) \quad (10)$$

$$Q_2 = \frac{1}{1+\beta}\left(\frac{\beta}{1+2\beta}P_1 + \frac{1+\beta}{1+2\beta}P_2\right) + \frac{\beta}{1+\beta}\left(\frac{2\beta/3}{1+2\beta}P_0 + \frac{1+4\beta/3}{1+2\beta}P_3\right) \quad (11)$$

To compute Q_1, first compute linear combination $R_{1,1}$ of P_1 and P_2, and linear combination $R_{1,2}$ of P_0 and P_3 as follows:

$$R_{1,1} = \frac{1+\beta}{1+2\beta}P_1 + \frac{\beta}{1+2\beta}P_2$$

$$R_{1,2} = \frac{1+4\beta/3}{1+2\beta}P_0 + \frac{2\beta/3}{1+2\beta}P_3$$

then compute the linear combination of $R_{1,1}$ and $R_{1,2}$ to get Q_1

$$Q_1 = \frac{1}{1+\beta}R_{1,1} + \frac{\beta}{1+\beta}R_{1,2}.$$

Similarly, to compute Q_2, first compute linear combination $R_{2,1}$ of P_1 and P_2, and linear combination $R_{2,2}$ of P_0 and P_3 as follows:

$$R_{2,1} = \frac{\beta}{1+2\beta}P_1 + \frac{1+\beta}{1+2\beta}P_2;$$

$$R_{2,2} = \frac{2\beta/3}{1+2\beta}P_0 + \frac{1+4\beta/3}{1+2\beta}P_3,$$

then compute the linear combination of $R_{2,1}$ and $R_{2,2}$ to get Q_2

$$Q_2 = \frac{1}{1+\beta}R_{2,1} + \frac{\beta}{1+\beta}R_{2,2}.$$

The relationship between these points is shown in Figure 4.

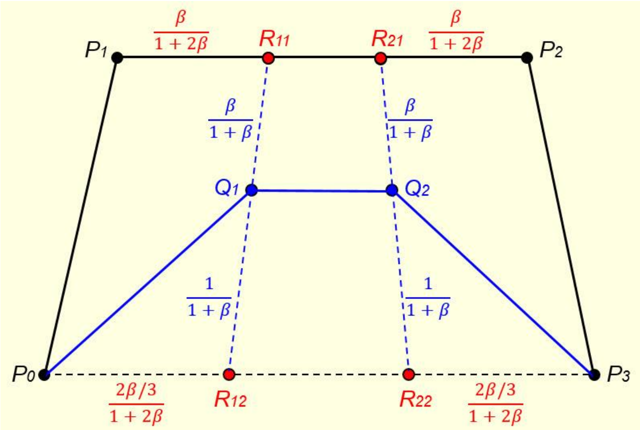

Figure 4. Relationship between control points of a cubic Beta-Bezier curve and its Bezier control points.

It is clear now how the shape parameter β applies a dripping force on the curve. Note that Q_1 and Q_2 move from points on the control polygon leg $\overline{P_1P_2}$ (when $\beta = 0$, $Q_1 = P_1$, $Q_2 = P_2$) to points on the control polygon leg $\overline{P_0P_3}$ (when $\beta = +\infty$, $Q_1 = \frac{2}{3}P_0 + \frac{1}{3}P_3$, $Q_2 = \frac{1}{3}P_0 + \frac{2}{3}P_3$)。 Therefore, according to (7), the curve will be pushed toward the base line segment $\overline{P_0P_3}$ when β becomes larger.

(vi) Beta-Bezier curves have a de Casteljau-like algorithm [3].

For any number t between 0 and 1, one can compute the value of a Bezier curve segment $C(t)$ of degree n as defined in (4) using the classic de Casteljau algorithm [4; 5; 8]:

for i from 0 to n do
  $P_i^0 = P_i;$
for i from 1 to n do
  for j from i to n do
    $P_j^i = (1-t) * P_{j-1}^{i-1} + t * P_j^{i-1};$

Figure 5. de Casteljau algorithm for a Bezier curve.

The value of $C(t)$ is the value contained in P_n^n when the computation process stops. A chart that illustrates the computation process for a cubic Bezier curve segment is shown in Figure 6.

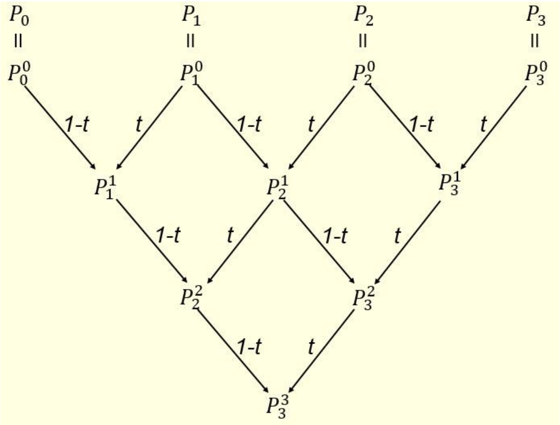

Figure 6. Computation flow of the de Casteljau algorithm for a cubic Bezier curve.

For a Beta-Bezier curve segment defined using the basis functions described in (1), a de Casteljau-like algorithm is given in [3]. By replacing $1/\lambda$ with β in the computation process, we get the following de Casteljau-like algorithm for Beta-Bezier curves defined in (2) and (3):

for i from 0 to n do
  $P_i^0 = P_i;$
for i from 1 to n do
  for j from i to n do
    $P_j^i = (\frac{1-t+(n-j)*\beta}{1+(n-i)*\beta}) * P_{j-1}^{i-1} + \left(\frac{t+(j-i)*\beta}{1+(n-i)*\beta}\right) * P_j^{i-1};$

Figure 7. de Casteljau algorithm for a Beta-Bezier curve.

The value of $C(t;\beta)$ is the value contained in P_n^n when the computation process stops. A chart that illustrates the computation process for a cubic Beta Bezier curve segment is shown in Figure 8.

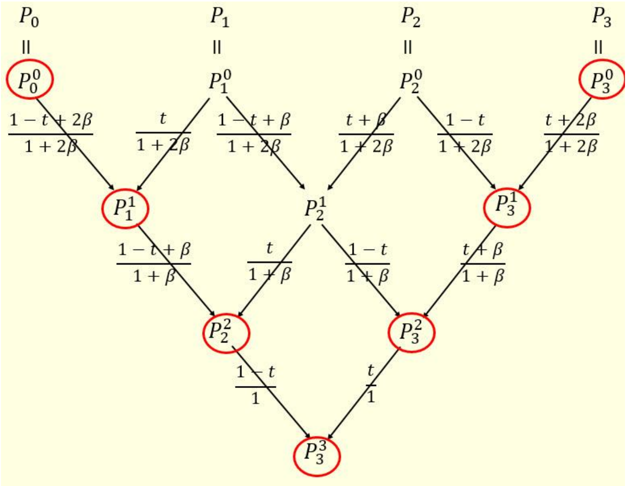

Figure 8. Computation flow of the de Casteljau algorithm for a cubic Beta-Bezier curve.

When $\beta = 0$ the de Casteljau-like algorithm for Beta-Bezier curves reduces to the de Casteljau algorithm for Bezier curves. However, unlike the de Casteljau algorithm for Bezier curves, the de Casteljau algorithm for Beta-Bezier curves cannot be used in the recursive subdivision process of a Beta-Bezier curve. Recursive subdivision technique for Beta-Bezier curves is shown in Section 4.

3 SMOOTHNESS CONDITIONS BETWEEN ADJACENT CURVE SEGMENTS

Two cubic Beta-Bezier curve segments can be joined together with C^0-, C^1- or C^2-continuity.

Note that the first derivative of a cubic Beta-Bezier curve is

$$C'(t;\beta) = \frac{3}{1+2\beta}[(P_1 - P_0)B_{2,0}(t;\beta) + (P_2 - P_1)B_{2,1}(t;\beta) + (P_3 - P_2)B_{2,2}(t;\beta)] + \frac{3\beta}{(1+\beta)(1+2\beta)}[(P_2 - P_0)B_{1,0}(t;\beta) + (P_3 - P_1)B_{1,1}(t;\beta)] + \frac{2\beta^2}{(1+\beta)(1+2\beta)}(P_3 - P_0) \quad (12)$$

and the second derivative of a cubic Beta-Bezier curve is

$$C''(t;\beta)=\frac{6}{(1+\beta)(1+2\beta)}[(P_2-2P_1+P_0)B_{1,0}(t;\beta)+(P_3-2P_2+P_1)B_{1,1}(t;\beta)]$$
$$+\frac{6\beta}{(1+\beta)(1+2\beta)}(P_3-P_2-P_1+P_0) \tag{13}$$

where $B_{2,i}(t;\ \beta), i=0,1,2$, and $B_{1,j}(t;\ \beta), j=0,1$, are basis functions of Beta-Bezier curves of degree 2 and degree 1, respectively.

Hence, for two adjacent, cubic Beta-Bezier curve segments $C_1(t;\ \beta_1)$ and $C_2(t;\ \beta_2)$ with control points $\{P_{1,0},P_{1,1},P_{1,2},P_{1,3}\}$ and $\{P_{2,0},P_{2,1},P_{2,2},P_{2,3}\}$ and shape parameters β_1 and β_2, respectively, to have C^1-continuity at the joint, we must have

$$C_1(1;\ \beta_1)=C_2(0;\ \beta_2);\qquad C_1'(1;\ \beta_1)=C_2'(0;\ \beta_2)$$

or

$$P_{1,3}=P_{2,0} \tag{14}$$

$$3(P_{1,3}-P_{1,2})+\frac{3\beta_1}{1+\beta_1}(P_{1,3}-P_{1,1})+\frac{2\beta_1^2}{1+\beta_1}(P_{1,3}-P_{1,0})$$
$$=3(P_{2,1}-P_{2,0})+\frac{3\beta_2}{1+\beta_2}(P_{2,2}-P_{2,0})+\frac{2\beta_2^2}{1+\beta_2}(P_{2,3}-P_{2,0}) \tag{15}$$

When $\beta_1=\beta_2=0$, (15) reduces to $P_{1,3}-P_{1,2}=P_{2,1}-P_{2,0}$, C^1-continuity condition for two adjacent, cubic Bezier curve segments at their joint.

When $\beta_i\neq 0, i=1{,}2$, C^1-continuity depends not only on the last control polygon leg of the first curve and first control polygon leg of the second curve, but also on vectors connecting the joint with the other vertices of the control polygons. Actually, the base line segments have the biggest impact on the continuity condition when β is large (see Figure 9).

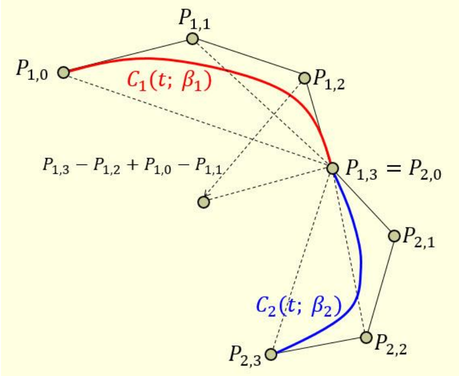

Figure 9. Connecting two cubic Beta-Bezier curve segments.

For $C_1(t;\ \beta_1)$ and $C_2(t;\ \beta_2)$ to have C^2-continuity at the joint, in addition to (14) and (15), we must also have $C_1''(1;\ \beta_1)=C_2''(0;\ \beta_2)$, that is,

$$\frac{1}{(1+\beta_1)(1+2\beta_1)}(P_{1,3}-2P_{1,2}+P_{1,1})+\frac{\beta_1}{(1+\beta_1)(1+2\beta_1)}(P_{1,3}-P_{1,2}-P_{1,1}+P_{1,0})$$
$$=\frac{1}{(1+\beta_2)(1+2\beta_2)}(P_{2,2}-2P_{2,1}+P_{2,0})+\frac{\beta_2}{(1+\beta_2)(1+2\beta_2)}(P_{2,3}-P_{2,2}-P_{2,1}+P_{2,0}) \tag{16}$$

When $\beta_1 = \beta_2 = 0$, (16) reduces to $P_{1,3} - 2P_{1,2} + P_{1,1} = P_{2,2} - 2P_{2,1} + P_{2,0}$, C^2-continuity condition of two adjacent, cubic Bezier curve segments at their joint.

When $\beta_i \neq 0, i = 1,2$, C^2-continuity depends also on $P_{1,3} - P_{1,2} - P_{1,1} + P_{1,0}$ and $P_{2,3} - P_{2,2} - P_{2,1} + P_{2,0}$ (see Figure 9). Since these vectors determine the curvature of the curves at the joint, when β is large, (16) indicates $C_1(t;\ \beta_1)$ and $C_2(t;\ \beta_2)$ have small curvature at the joint.

4 RECURSIVE SUBDIVISION

In this section we show that given a cubic Beta-Bezier curve segment $C(t;\ \beta)$ as defined in (6) and a $0 < t_0 < 1$, by computing the value of $C(t_0;\ \beta)$ the curve is divided into two Beta-Bezier curve segments at $C(t_0;\ \beta)$, each with its own control points. Therefore, one can perform recursive subdivision on cubic Beta-Bezier curves.

More specifically, when one uses the de Casteljau algorithm (Figure 7) to compute the value of $C(t_0;\ \beta)$ for a given $0 < t_0 < 1$, the value one gets in P_3^3 divides the curve segment into two portions:

$$C_1 \equiv \{\, C(t;\ \beta) \mid 0 \le t \le t_0\} \qquad (17)$$
$$C_2 \equiv \{\, C(t;\ \beta) \mid t_0 \le t \le 1\}$$

C_1 and C_2 are each a cubic Beta-Bezier curve with shape parameter β. However, unlike the situation for classic cubic Bezier curve segments, the control points of C_1 and C_2 are not $\{P_0^0, P_1^1, P_2^2, P_3^3\}$ and $\{P_3^3, P_3^2, P_3^1, P_3^0\}$ (the circled points in Figure 8), but $\{Q_0^0, Q_1^1, Q_2^2, Q_3^3\}$ and $\{Q_3^3, Q_3^2, Q_3^1, Q_3^0\}$. These control points are computed as follows.

First, $C(t;\ \beta)$ is converted to a cubic Bezier curve $\bar{C}(t)$ with control points P_0, Q_1, Q_2 and P_3 as follows:

$$\bar{C}(t) = (1-t)^3 P_0 + 3t(1-t)^2 + 3t^2(1-t)Q_2 + t^3 P_3$$

where $0 \le t \le 1$ and Q_1 and Q_2 are defined in (8) and (9), but should be computed using the process shown in Figure 4. For the convenience of subsequent indexing, we will use Q_0^0, Q_1^0, Q_2^0 and Q_3^0 to represent the control points of $\bar{C}(t)$, that is, $Q_0^0 \equiv P_0$, $Q_1^0 \equiv Q_1$, $Q_2^0 \equiv Q_2$, $Q_3^0 \equiv P_3$. A subdivision step is then performed on $\bar{C}(t)$ by applying the classic de Casteljau algorithm (Figure 5) on the control point set $\{Q_0^0, Q_1^0, Q_2^0, Q_3^0\}$ for the given t_0. The computation process is shown in Figure 10.

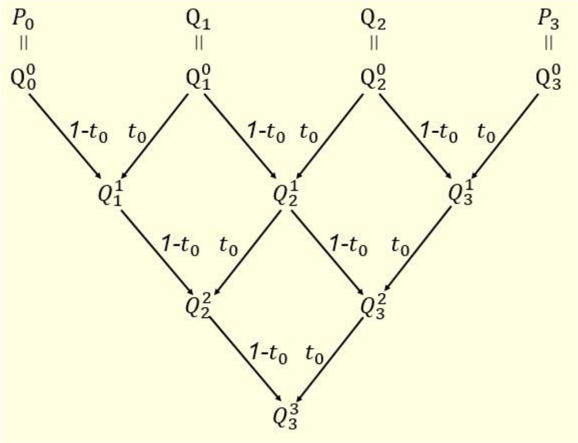

Figure 10. Computing the value of $C(t;\ \beta)$ at t_0 when represented as a cubic Bezier curve.

The subdivision step splits $\bar{C}(t)$ at $\bar{C}(t_0) = Q_3^3$ into two cubic Bezier curve segments $\bar{C}_1(t)$ and $\bar{C}_2(t)$ with control point sets $\{Q_0^0, Q_1^1, Q_2^2, Q_3^3\}$ and $\{Q_3^3, Q_3^2, Q_3^1, Q_3^0\}$, respectively, that is,

$$\bar{C}_1(s) = (1-s)^3 Q_0^0 + 3s(1-s)^2 Q_1^1 + 3s^2(1-s) Q_2^2 + s^3 Q_3^3, \qquad 0 \le s \le 1 \tag{18}$$

and

$$\bar{C}_2(s) = (1-s)^3 Q_3^3 + 3s(1-s)^2 Q_3^2 + 3s^2(1-s) Q_3^1 + s^3 Q_3^0, \qquad 0 \le s \le 1 \tag{19}$$

An example illustrating the situation for a shape parameter $\beta = 1$ and $t_0 = 1/2$ is shown in Figure 11.

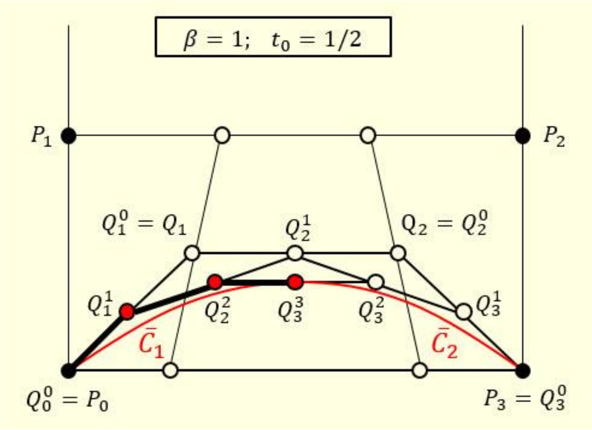

Figure 11. Subdivision of a cubic Beta-Bezier curve segment with shape parameter $\beta = 1$ at $t_0 = 1/2$.

To show that $\{Q_0^0, Q_1^0, Q_2^0, Q_3^0\}$ is the control point set of C_1 defined in (17), it is sufficient to show that $\bar{C}_1(s)$ defined in (18) is the same as C_1. Note that from Figure 10 we can easily derive that

$$Q_1^1 = (1-t_0)P_0 + t_0 Q_1$$
$$Q_2^2 = (1-t_0)^2 P_0 + 2t_0(1-t_0)Q_1 + {t_0}^2 Q_2$$
$$Q_3^3 = (1-t_0)^3 P_0 + 3t_0(1-t_0)^2 Q_1 + 3{t_0}^2(1-t_0)Q_2 + {t_0}^3 P_3$$

By substituting these expressions into $\bar{C}_1(s)$ and through straightforward algebra, one gets that

$$\bar{C}_1(s) = (1-st_0)^3 P_0 + 3(st_0)(1-st_0)^2 Q_1 \\ +3(st_0)^2(1-st_0)Q_2 + (st_0)^3 P_3, \qquad 0 \le s \le 1$$

Then by substituting the expressions for Q_1 and Q_2 from (10) and (11) into the above equation and through straightforward algebra, we would have

$$\bar{C}_1(s) = \frac{(1-st_0)(1-st_0+\beta)(1-st_0+2\beta)}{(1+\beta)(1+2\beta)}P_0 + \frac{3(st_0)(1-st_0)(1-st_0+\beta)}{(1+\beta)(1+2\beta)}P_1 \\ + \frac{3(st_0)(1-st_0)(st_0+\beta)}{(1+\beta)(1+2\beta)}P_2 + \frac{(st_0)(st_0+\beta)(st_0+2\beta)}{(1+\beta)(1+2\beta)}P_3 \\ = C(st_0;\, \beta), \qquad 0 \le s \le 1$$

Note that $\{C(st_0;\,\beta) | 0 \le s \le 1\} = C_1$. Hence $\bar{C}_1(s)$ is indeed the same as C_1 and therefore $\{Q_0^0, Q_1^1, Q_2^2, Q_3^3\}$ is indeed the control point set of C_1. The proof for the C_2 case is similar and, therefore, will not be shown here.

The above proof also explains why $\{P_0^0, P_1^1, P_2^2, P_3^3\}$ is not the control point set of C_1. Note that Q_1^1 as a combination of P_0 and Q_1 involves all of the control points of $C(t;\,\beta)$ and yet P_1^1 only involves P_0 and P_1. Therefore, P_1^1 cannot be the second control point of C_1. Likewise, Q_2^2 also involves all the control points of $C(t;\,\beta)$ while P_2^2 only involves P_0, P_1 and P_2. Hence P_2^2 cannot be the third control point of C_1 either.

5 INTERPOLATION USING COMPOSITE CUBIC BETA-BEZIER CURVES

Given a set of data points D_0, D_1, D_2, $\cdots$, D_n, one can construct a composite cubic Beta-Bezier curve to interpolate these points. The curve is C^2- continuous.

In the case the data set is closed ($D_n = D_0$) (see Figure 12 for an example with $n = 5$), the composite cubic Beta-Bezier curve has n segments and each segment $C_k(t;\,\beta_k)$, $1 \le k \le n$, is a cubic Beta-Bezier curve with control point set $\{P_{k,0}, P_{k,1}, P_{k,2}, P_{k,3}\}$ and shape parameter β_k. $C_k(t;\,\beta_k)$ interpolates D_{k-1} and D_k at its start point and end point, that is, $P_{k,0} = D_{k-1}$ and $P_{k,3} = D_k$. Hence, for each segment $C_k(t;\,\beta_k)$, only two control points, $P_{k,1}$ and $P_{k,2}$ have to be constructed. Totally, we have $2n$ unknowns for the construction of such a C^2 - continuous composite cubic Beta-Bezier curve. These unknowns can be found using C^1- and C^2- continuity conditions at the data points.

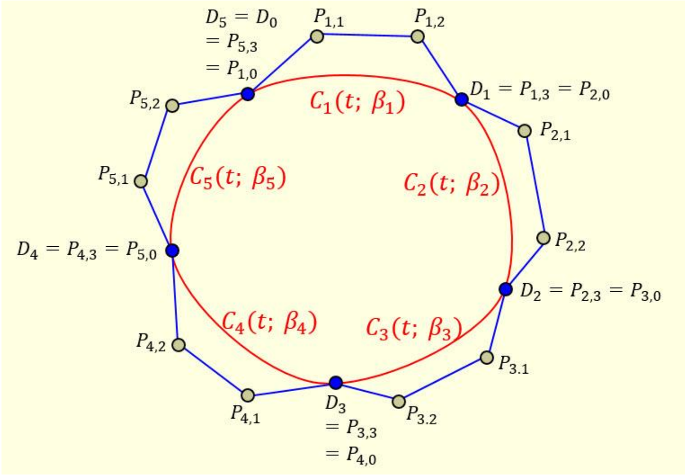

Figure 12. Interpolation using a composite cubic Beta-Bezier curve.

Note that for $k = 1, 2, \ldots, n$, we have

$$D_k = P_{k,3} = P_{k+1.0} \qquad (k + 1\ mod\ n)$$

If β_k, $k = 1, 2, \ldots, n$, are given then from (15) we have

$$\alpha_k P_{k,1} + P_{k,2} + P_{k+1,1} + \alpha_{k+1} P_{k+1,2} = -\frac{2\alpha_k\beta_k}{3} D_{k-1} + [2 + \alpha_k + \alpha_{k+1} + \frac{2\alpha_k\beta_k}{3} + \frac{2\alpha_{k+1}\beta_{k+1}}{3}]D_k - \frac{2\alpha_{k+1}\beta_{k+1}}{3} D_{k+1} \qquad (20)$$

where $1 \le k \le n$ and $\alpha_i \equiv \beta_i/(1 + \beta_i)$, and from (16) we have

$$[(1 - \beta_k)\gamma_k]P_{k,1} - [(2 + \beta_k)\gamma_k]P_{k,2} + [(2 + \beta_{k+1})\gamma_{k+1}]P_{k+1,1} - [(1 - \beta_{k+1})\gamma_{k+1}]P_{k+1,2} = -(\beta_k\gamma_k)D_{k-1} + \left(-\frac{1}{1+2\beta_k} + \frac{1}{1+2\beta_{k+1}}\right)D_k + (\beta_{k+1}\gamma_{k+1})D_{k+1} \qquad (21)$$

where $1 \le k \le n$ and $\gamma_i \equiv 1//(1 + \beta_i)/(1 + 2\beta_i)$.

By solving this system of 2n equations for $P_{k,1}$ and $P_{k,2}$, $k = 1, 2, \ldots, n$, we get the remaining control points for the construction of the C^2- continuous, composite cubic Beta-Bezier curve that interpolates the given data points. However, assigning different shape parameters to a large number of segments is a tedious work. Besides, for many applications it is the overall shape that is important especially when the size of each individual segment is relatively small. A more appropriate approach is to use a global shape parameter β for all the segments of the interpolating composite curve initially and then adjust the value of β to adjust the shape of the interpolating composite curve after it is constructed. In such a case, (20) and (21) become of the following forms:

$$\alpha P_{k,1} + P_{k,2} + P_{k+1,1} + \alpha P_{k+1,2} = -\frac{2\alpha\beta}{3} D_{k-1} + 2(1 + \alpha + \frac{2\alpha\beta}{3})D_k - \frac{2\alpha\beta}{3} D_{k+1} \qquad (22)$$

where $1 \le k \le n$, $\alpha = \beta/(1 + \beta)$ and β is a given global shape parameter,

$$[(1 - \beta)\gamma]P_{k,1} - [(2 + \beta)\gamma]P_{k,2} + [(2 + \beta)\gamma]P_{k+1,1} - [(1 - \beta)\gamma]P_{k+1,2}$$

$$= -(\beta\gamma)D_{k-1} + (\beta\gamma)D_{k+1}, \tag{23}$$

where $1 \le k \le n$ and $\gamma = 1/(1+\beta)(1+2\beta)$. The example shown in Figure 12 is construction this way with $\beta = 1$. Note that in this case once the above system is solved and the interpolating composite cubic Beta-Bezier curve is computed, to change the shape of the curve, we simply change the value of β and compute the new shape of each segment using the original control points, we don't need to solve the above system again.

In the above system, if the initial value of the global shape parameter β is set to zero, then we have the following system of equations;

$$P_{k,2} + P_{k+1,1} = 2D_k,\ 1 \le k \le n \tag{24}$$

$$P_{k,1} - 2P_{k,2} + 2P_{k+1,1} - P_{k+1,2} = 0, \quad 1 \le k \le n \tag{25}$$

This is the system we need to solve to get a composite cubic Bezier curve to interpolate the given data points. We cannot change the shape of the curve once it is constructed unless we change the locations of some of the data points.

For an open interpolating curve, we only have 2(n-1) conditions in any of the above systems. To get two extra conditions, one approach is to set the second derivatives at D_0 and D_n to be zero.

6 REPRESENTATION CONVERSION

A cubic Beta-Bezier curve segment can be represented as a cubic B-spline curve segment.
Given a cubic Beta-Bezier curve $C(t;\ \beta)$ with control point set $\{P_0, P_1, P_2, P_3\}$ and shape parameter β,

first convert it to a cubic Bezier curve as the one shown in (7) with Q_1 and Q_2 being defined as in

(8) and (9), see Figure 13 for an illustration. We then compute $\bar{P}_0$, $\bar{P}_1$, $\bar{P}_2$ and $\bar{P}_3$ as follows:

$$\begin{aligned} &\bar{P}_1 = Q_1 + (Q_1 - Q_2); \qquad &&\bar{P}_2 = Q_2 + (Q_2 - Q_1) \\ &A_1 = P_0 + (P_0 - Q_1); &&A_2 = P_3 + (P_3 - Q_2) \\ &\bar{P}_0 = A_1 + 2(A_1 - \bar{P}_1); &&\bar{P}_3 = A_2 + 2(A_2 - \bar{P}_2) \end{aligned} \tag{26}$$

If we define a cubic B-spline curve segment $CB(t)$ using $\bar{P}_0$, $\bar{P}_1$, $\bar{P}_2$ and $\bar{P}_3$ as its control points as

follows:

$$CB(t) = \frac{(1-t)^3}{6}\bar{P}_0 + \frac{4-6t^2+3t^3}{6}\bar{P}_1 + \frac{1+3t+3t^2-3t^3}{6}\bar{P}_2 + \frac{t^3}{6}\bar{P}_3 \tag{27}$$

where $0 \le t \le 1$ then by substituting the expressions defined in (26) into (27) for $\bar{P}_0$, $\bar{P}_1$, $\bar{P}_2$ and

$\bar{P}_3$, with Q_1 and Q_2 being replaced with the expressions defined in (8) and (9), it can be proved

through straight forward algebra that $CB(t) = C(t;\ \beta)$, that is, the cubic B-spline curve segment defined in (27) equals $C(t;\ \beta)$. Hence, a cubic Beta-Bezier curve segment can indeed be represented as a cubic B-spline curve segment as well.

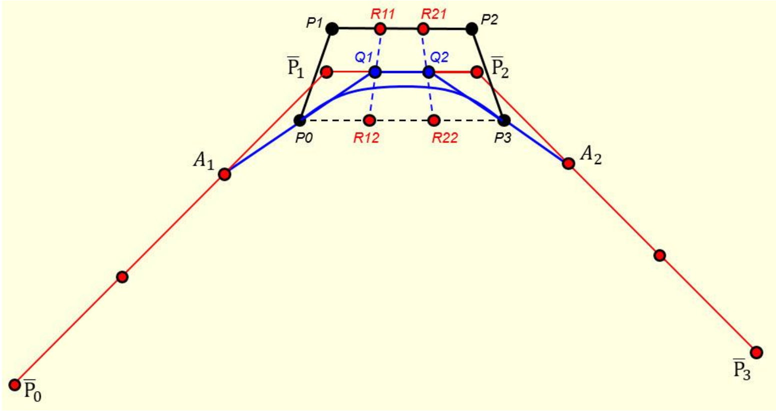

Figure 13. Representing a cubic Beta-Bezier curve segment as a cubic B-spline curve segment.

7 CONCLUDING REMARKS

This paper gives a new definition of Beta-Bezier curves. Like the original definition [3], the new definition also includes the classic Bezier curves as a special case. The main difference between the new definition and the original definition is on the role of the shape parameter. In the new definition, the shape parameter is the inverse of the shape parameter used in the original definition.

With the new definition, properties of Beta-Bezier curves are easier to study. It shows that Beta-Bezier curves not only have all the basic properties of Bezier curves such as convex hull property, recursive subdivision, B-spline conversion and C^2 interpolation, but also the capability of modifying the shape a Bezier curve segment or a C^2-continuous, composite cubic Bezier curve without changing the control points of the curve. This is because in the cubic case a Beta-Bezier curve is actually also a Bezier curve. Consequently, we have a curve design technique more general than Bezier curves. Since C^2-continuous, composite cubic Bezier curves are equivalent to uniform B-spline curves, this means the new curve design technique is more general than uniform B-spline curves as well.

Future works in this direction include the study of Beta-Bezier surfaces, extending the Beta shape parameter concept into B-spline curves and surfaces, and subdivision surfaces as well.

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A APPENDICES

A.1 Convex Hull Property

Beta-Bezier curves satisfy the convex hull property. The proof is done by induction. However, the general proof is too lengthy to be presented here. We will show that if the case is true for degree 3, then we would get the degree 4 case from the degree 3 case. The general case follows the same concept.

When degree = 3, we have

$$\boldsymbol{B_{3,0}(t;\ \beta) + B_{3,1}(t;\ \beta) + B_{3,2}(t;\ \beta) + B_{3,3}(t;\ \beta)}$$

$$= \frac{\boldsymbol{1}}{\boldsymbol{(1+\beta)(1+2\beta)}}[\boldsymbol{(1-t)(1-t+\beta)(1-t+2\beta) + 3(1-t)(1-t+\beta)t + 3(1-t)t(t+\beta)}$$
$$\boldsymbol{+t(t+\beta)(t+2\beta)}]$$

$$= \frac{\boldsymbol{1}}{\boldsymbol{(1+\beta)(1+2\beta)}}[\boldsymbol{(1-t)^3 + 3\beta(1-t)^2 + 2\beta^2(1-t) + 3t(1-t)^2 + 3\beta t(1-t) + 3t^2(1-t)}$$
$$\boldsymbol{+3\beta t(1-t) + t^3 + 3\beta t^2 + 2\beta^2 t}]$$

$$= \frac{\boldsymbol{1}}{\boldsymbol{(1+\beta)(1+2\beta)}}[\boldsymbol{1 + 3\beta + 2\beta^2}]$$

$$= \boldsymbol{1}$$

Hence degree 3 case is true.

When degree = 4, we have

$$\boldsymbol{B_{4,0}(t;\ \beta) + B_{4,1}(t;\ \beta) + B_{4,2}(t;\ \beta) + B_{4,3}(t;\ \beta) + B_{4,4}(t;\ \beta)}$$

$$= \frac{\boldsymbol{1}}{\boldsymbol{(1+\beta)(1+2\beta)(1+3\beta)}}[\boldsymbol{(1-t)(1-t+\beta)(1-t+2\beta)}$$

$$
\begin{aligned}
&* (1-t+3\beta) + 4(1-t)(1-t+\beta)(1-t+2\beta)t \\
&+6(1-t)(1-t+\beta)t(t+\beta) + 4(1-t)t(t+\beta)(t+2\beta) \\
&+ t(t+\beta)(t+2\beta)(t+3\beta)] \\
&= \frac{1}{(1+\beta)(1+2\beta)(1+3\beta)}[-(1-t)(1-t+\beta)(1-t+2\beta)t \\
&+(1-t)(1-t+\beta)(1-t+2\beta)(1+3\beta) + (1-t) \\
&* (1-t+\beta)(1-t+2\beta)t + 3(1-t)(1-t+\beta)t(t+\beta) \\
&+3(1-t)(1-t+\beta)t(t+\beta) + 4(1-t)t(t+\beta)(t+2\beta) \\
&-(1-t)t(t+\beta)(t+2\beta) + t(t+\beta)(t+2\beta)(1+3\beta)] \\
&= \frac{1}{(1+\beta)(1+2\beta)(1+3\beta)}[(1-t)(1-t+\beta)(1-t+2\beta) \\
&* (1+3\beta) + 3(1-t)(1-t+\beta)t(1+3\beta) \\
&+3(1-t)t(t+\beta)(1+3\beta) + t(t+\beta)(t+2\beta)(1+3\beta)] \\
&= \frac{1}{(1+\beta)(1+2\beta)}[(1-t)(1-t+\beta)(1-t+2\beta) + 3(1-t)(1-t+\beta)t + 3(1-t)t(t+\beta) \\
&+t(t+\beta)(t+2\beta)] \\
&= B_{3,0}(t;\ \beta) + B_{3,1}(t;\ \beta) + B_{3,2}(t;\ \beta) + B_{3,3}(t;\ \beta)
\end{aligned}
$$

Since degree 3 case equals 1, degree 4 case hence equals 1 too.